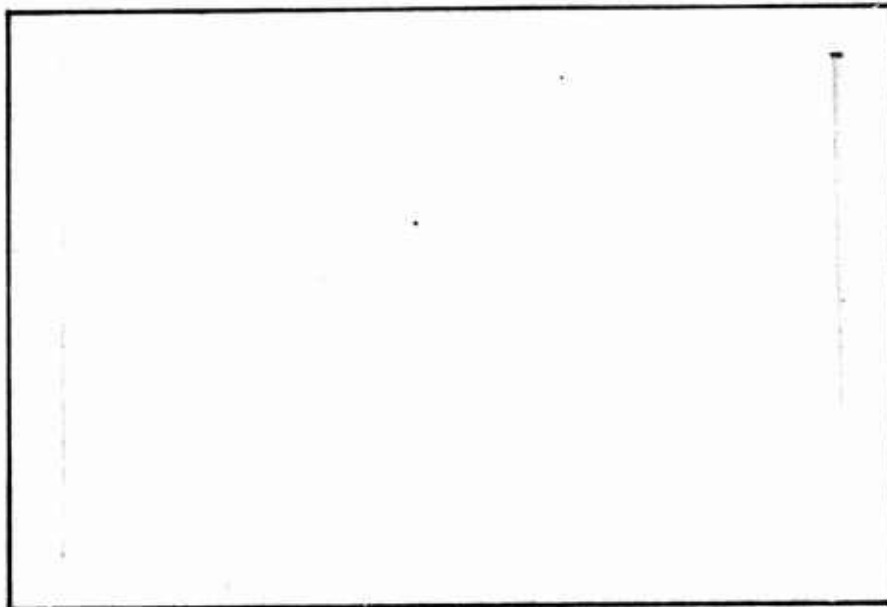


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APPROXIMATION TECHNIQUES AND
OPTIMAL DECISION MAKING FOR
STOCHASTIC LANCHESTER MODELS.

by

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APPROXIMATION TECHNIQUES AND OPTIMAL DECISION
MAKING FOR STOCHASTIC LANCHESTER MODELS

by

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ABSTRACT

This thesis extends the analysis of stochastic Lanchester models beyond the stage of mere modeling. To this end, a framework of statistical decision theory is superimposed on a simplified combat situation. The commander must make decisions about the amount of force he will commit to a combat in reference to a suitable cost and reward structure. Problems of both the one-stage and the multi-stage variety are studied.

The one-stage decision problem requires knowledge of the probability of victory and the expected number of survivors. A complete solution to this problem is given, based on the use of a martingale central limit theorem. The multi-stage decision problem requires the distribution of the force level configuration as a function of time. These distributions are approximated through the use of diffusion approximations. A two-stage problem is solved using these approximations and backward induction.

Some numerical studies are presented to provide empirical support for the accuracy and utility of the methods. The results of these studies are very encouraging, giving strong support to the efficacy of the proposed methodologies for the solution of decision problems. In addition, the diffusion approximation methodology provides an important contribution to the study of attrition processes of the Lanchester type in the continuous time setting.

Chapter 1

INTRODUCTION

1.1. Historical Background

"War is a matter of vital importance to the State; the province of life or death; the road to survival or ruin. It is mandatory that it be thoroughly studied." (Griffith, 1963.)

So begins the military classic The Art of War by Sun Tzu, perhaps the most ancient of surviving texts on the subject. In the twenty-four centuries following the appearance of Sun Tzu's treatise, innumerable philosophers and soldiers have written their own analyses of the art and science of warfare. Many of these works are very specific manuals which attempt to outline the techniques necessary to successfully practice the type of warfare prevalent at that time. Others seek to elucidate the basic principles of war. Yet all of these works have one common goal, to divine the secrets of waging war as successfully and cheaply as possible.

Prior to the Twentieth Century, the scientist, mathematician and engineer were seldom directly involved in the study of warfare. Although mathematics of one form or another had always played a role in military affairs, (for example, in estimating the time required for an army to march a certain distance or the angle at which to fire an artillery piece to achieve a desired range) that role was a very minor one. The two World Wars of this century elevated mathematics to a much more important role in the analysis of military problems.

1.2. Lanchester Theory

The World War of 1914-1918 saw the physical sciences begin to have an increasingly important impact on military affairs. It was also during this period that a pioneering effort at a mathematical analysis of warfare first appeared.

In a series of articles written for the journal Engineering, Frederick William Lanchester discussed the importance of aircraft in modern war. In the fifth of these articles (Lanchester, 1914), Lanchester sought to provide more general insights into the nature of combat, and to consider the conditions under which a force which is numerically inferior to its enemy might yet be victorious in battle.

Lanchester considered two opposing forces, Red and Blue whose initial force levels are X_0 and Y_0 respectively, and whose force levels at any time $t > 0$ are given by X_t and Y_t . The ability of Blue units to destroy Red units is characterized by a constant, a , known as an attrition coefficient, and that of Red to destroy Blue by a constant b . Lanchester's models are of two types, both in the form of a system of ordinary differential equations.

The first of Lanchester's models is described by the system of equations (1.2.1).

$$\frac{dx}{dt} = \begin{cases} -axy & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \quad \frac{dy}{dt} = \begin{cases} -bxy & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.2.1)$$

If the time dependence in these equations is eliminated, a

relationship, known as the state equation, results. The state equation for this model is

$$X_0 - X_t = \frac{a}{b}(Y_0 - Y_t). \quad (1.2.2)$$

Since (1.2.2) is a linear equation, Lanchester named model (1.2.1) the "Linear Law." Lanchester looked upon this model as descriptive of a combat situation which is composed of many individual duels, and for this reason he referred to it as a model of "ancient" combat. The implications of this model may be illustrated by an example (from Taylor (1975)). Let the initial Blue force level be $Y_0 = 100$ while its final level is $Y_f = 0$. If $a = b$, then for the given values of X_0 , the corresponding values of X_f and $X_0 - X_f$ are given in the following table:

Table (1.2.1) - The Linear Law

X_0 :	100	150	200	250	300	500
X_f :	0	50	100	150	200	400
$X_0 - X_f$:	100	100	100	100	100	100

Thus, regardless of the initial force employed by Red, its casualties remain the same.

Lanchester's second model is embodied in equations (1.2.3).

$$\frac{dx}{dt} = \begin{cases} -ay & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \quad \frac{dy}{dt} = \begin{cases} -bx & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (1.2.3)$$

The state equation corresponding to this model is

$$x_o^2 - x_t^2 = \frac{a}{b}(y_o^2 - y_t^2). \quad (1.2.4)$$

Equation (1.2.4) is quadratic, and this led Lanchester to name the model (1.2.3) the "Square Law." Lanchester considered this model to be descriptive of a combat situation in which several individuals could concentrate their strength against a few. Since such concentrated effort is a characteristic of fire weapons, he referred to the Square Law as a model of "modern" combat.

Under the same initial conditions as proposed for Table (1.2.1), the Square Law model yields Table (1.2.2).

Table (1.2.2) - The Square Law

x_o :	100	150	200	250	300	500
x_f :	0	112	173	229	283	490
$x_o - x_f$:	100	38	27	21	17	10

In this case there is a dramatic reduction in Red casualties as the initial force level increases, indicating that casualties may be minimized by deploying maximum force on the battlefield.

The sorts of results indicated above led Lanchester to conclude that, while concentration* is of minor importance under

* The term concentration as used here and in the following pages does not necessarily imply a close physical proximity; rather it means the commitment of more force to the actual combat.

conditions of "ancient" warfare, the nature of modern war requires maximum deployment of strength on the field of battle. Lanchester propounded these results as mathematical support for the "Principle of Concentration." According to this principle, a numerically superior force should seek to remain concentrated to take full advantage of its numbers. An inferior force, on the other hand, must induce its enemy to divide his strength so that numerical advantage may be achieved against each portion of the enemy force in turn despite the overall inferiority.

1.3. Limitations of Lanchester's Models

As can be seen from the discussion above, the Lanchester attrition laws in their original form are based on the premise that units on each side are homogeneous, and their differences may be summarized by the attrition coefficients. These and other simplistic assumptions dictate that Lanchester's models must remain crude approximations to the highly complex dynamics of modern combat. There is little evidence that Lanchester intended his Linear and Square Laws to be detailed combat models, and they have many shortcomings which limit their ability to fill such a role. Some of these shortcomings are outlined by Taylor (1975). The more important areas in which Lanchester's Laws make simplifying assumptions are:

1. The forces are considered homogeneous and the attrition structure symmetric. There are many combat situations of interest in which neither assumption is valid.
2. The attrition rate coefficients are constant. There is no provision for the possibility of changes in weapon or unit effectiveness due to a variety of physical, temporal and psychological factors.
3. The effect of friendly fire on suppressing that of the enemy is not explicitly taken into account. This attitude discounts a very important facet of fire combat.
4. The models are deterministic rather than stochastic. As such, they fail to reflect the wide variability of

actual combat. For given initial conditions, the results of the models are always the same.

5. There is no provision for control and decision making such as the choice of what force level to employ, when to withdraw, etc. Because of these lacks, important tactical and strategic questions such as the effectiveness of command, control and communication systems cannot be studied directly.
6. Complete information is assumed to be available about the enemy's strength and combat abilities. Typically, there is a fairly high degree of uncertainty about these topics present in actual combat and conclusions must be reached with these uncertainties in mind.

1.4. Literature Survey

In the more than sixty years since their inception, Lanchester's basic models have been combined and extended in many ways, always with the hope of creating more detailed and realistic models which could successfully address some of the problems outlined above. Brackney (1959) examined non-symmetric attrition structures which combined elements of both Lanchester Laws. An example of this type of attrition is a model of an assault on a prepared, concealed position. In this model the attacker can find few point targets and so must spread his fire over the suspected area of enemy occupation. The defender, on the other hand, may execute aimed fire at the more exposed troops of the advancing attacker. Accepting the Linear Law as a reasonable model for area fire, and the Square Law as being a reasonable one for point fire, Brackney's equations for this type of attrition structure are of the form: $x'(t) = -ay$, $y'(t) = -bxy$ with state equation given by

$$y_0 - y_t = \frac{b}{a}(x_0^2 - x_t^2)$$

where X represents the attacker and Y the defender.

Deitchman (1962) employed the same concept as Brackney to model ambushes which might take place in a guerilla war. Schaffer (1968) developed a more general and detailed model of guerilla warfare. This model allowed for the use of supporting fires, such as artillery, and included provisions for losses due to surrenders and desertions during combat.

Helmbold (1965, 1966) introduced a general attrition structure of which Lanchester's Linear and Square Laws represent special cases. The general form for these models is:

$$x'(t) = -ah(x/y)y, \quad y'(t) = -bg(y/x)x.$$

The functions h and g which Helmbold examined in most detail were of the form $h(u) = g(u) = u^c$. Such a model leads to state equations of the type:

$$b(x_0^d - x_t^d) = a(y_0^d - y_t^d) \quad \text{where } d = 2c.$$

These types of models, for certain values of d (such as $d = 1/2$) actually penalize a numerically superior force if the disparity between forces is extremely great. This allows the introduction of economy of force considerations.

Rashevsky (1949), while retaining the square law attrition structure for both sides, considered changes in losses due to the speed of retreat of one side. The effects of force separation and other spatial variables were considered by Weiss (1957). Taylor (1971, 1974, 1974a, 1975) has done extensive work with time variable attrition rate coefficients in an attempt to model changes in weapon effectiveness over time.

Schaffer (1968), in his models of guerilla warfare, introduced, explicitly, the suppressive effects of supporting weapons. He also discussed the effects the length of an engagement might have on the actual attrition structure of a combat, outlining conditions under which area fire, or a linear type model, might change to point fire, or a quadratic type model.

All of the models discussed above are basically deterministic, and the authors of each attempt to detail physical assumptions which would give rise to their particular attrition structures. There have also been other such attempts, notably by Schreiber (1964); however, most have suffered from some arbitrariness and a general lack of rigorous construction. The same problems are visible in much of the early work on stochastic Lanchester models. Such stochastic models are generally assumed to be bivariate or multivariate Markov chains with time homogeneous transition rates (see Chapter 2). The difficulty with most of the models prior to Karr (1974) (see discussion below) was that they were usually developed directly from existing deterministic models, reversing the more usual process of deriving deterministic approximations to the more general stochastic systems. Because of this, there was seldom any attempt to make a formal derivation of the mathematical models from underlying physical assumptions.

Some of the earliest investigations into stochastic versions of Lanchester's Laws were made by Brown (1955, 1963) who developed expressions for the marginal force level distributions at time t and the probability of winning (i.e. completely destroying the enemy force) for either side. Although modern computing facilities may allow relatively easy calculation of these probabilities, the complexity of the expressions gives them only limited practical value in an extended analysis.

Smith (1965), expanding on Brackney (1959) and Deitchman (1962), developed probability distributions for the number of

survivors of a combat action, employing techniques of the calculus of variations in his derivation. Again, his expressions proved somewhat cumbersome to work with analytically. Kisi and Hirose (1966) further considered various models from a stochastic point of view, and developed the probability of victory in a Deitchman-type ambush engagement.

Brooks (1965) introduced the concept of "stochastically determined" large-scale battle models. A model is called stochastically determined if its overall results, such as the total casualties to each side, show little variance relative to the initial forces employed. The usual stochastic versions of Lanchester's Linear and Square Laws were shown to be stochastically determined. The significance of stochastic determinism is that it may allow the use of a fairly crude deterministic approximation to get rough estimates of overall results in large scale stochastic models without too serious an error.

In more recent developments, Grubbs and Shuford (1973) introduced a new and most interesting stochastic formulation of Lanchester's theory. They contended that the true random variable of interest in combat models is the "time to kill" of either side. Their approach centers around reliability theory, and they make heavy use of the flexibility inherent in the Weibull distribution. This approach seems to have promise in the field of small unit combat actions.

Karr (1974) provided the first rigorous derivations from sets of axioms for many Lanchester-type models. His paper is

a compendium of models both simple, as with Lanchester's Linear and Square Laws, and complex, as with models using non-homogeneous units and multiple kill capabilities for weapons. In all cases, Karr begins with a very specific set of assumptions about the physical nature of a combat situation. From these assumptions he then derives a time homogeneous Markov model and extracts from it the corresponding deterministic equations. This work provides a firm foundation for analysis and criticism of a model, allowing discussion to center around the appropriateness of the underlying physical assumptions rather than that of its mathematical form.

Another important contribution to the analysis of stochastic Lanchester-type models was made by Watson (1976). Watson introduced the idea of employing martingale methods to derive approximations for such quantities as the probability of victory and the distribution of the number of survivors in the case of the Lanchester Square Law. Although this idea is applicable to many other cases as well, Watson's paper itself is incomplete since he employs an unstated martingale central limit theorem and ignores various other difficulties which will be discussed in more detail in Chapter 3. Despite these weaknesses, Watson's results appear to be valid and important.

The inherent two sided nature of most combat situations makes the area rich for game theoretic investigations. Most of the modelling in this field has concentrated on a differential games approach (see for example Chattopadhyay (1969)). Weiss (1959) and Kawara (1973) examined the use of supporting fire

from such a point of view while Isbell and Marlow (1956) studied overall fire allocation policies. More recently Taylor (1972) expanded on the work of Isbell and Marlow and further investigated the area of tactical differential games (Taylor (1974b)).

Most of the existing work on optimal control and choice of tactical options has derived from the differential games approach. Two papers by Taylor (1973, 1974) examined optimal control of fire allocation over two types of enemy units, both of which are subject to a deterministic linear law attrition process. He discusses an optimal fire distribution policy with reference to the effects such factors as combatant objectives, battle termination criteria, attrition processes, and variable attrition rates may have on optimal choices. He employs control theory to prove his results, and he also discusses some of the implications in the fields of intelligence, command and control systems, and human decision making. Schreiber's note (1964) on the value of intelligence and command control also provides some insights into the effects of these two critical areas on combat situations.

Estimation of attrition parameters for both deterministic and stochastic models is another important area of investigation. Bonder (1967, 1970) considered techniques for estimating Lancaster coefficients in various models by using weapons system performance data. Barfoot (1969) attempted to improve on Bonder's original work by proposing to estimate attrition coefficients by the reciprocal of the average time to kill an opposing unit. This average time to kill was itself calculated using weapons

performance data, and allowed for possible corrections in firing procedure based on previous results. Clark (1968) employed maximum likelihood estimation based on results obtained from a detailed combat simulation. Rustagi and Srivastava (1968) considered the problem of parameter estimation in a Markov dependent model and employed maximum likelihood techniques. In a later paper, Rustagi and Laitinen (1970) dealt with estimation of moments for the same types of models.

The basic question of the reasonableness and applicability of Lanchester theory to actual combat has received attention from various sources. Engel (1954) analyzed casualty data from the battle for Iwo Jima. He found that Lanchester's Square Law seemed to give a reasonably good fit, although other Lanchester type models may also have been applicable. Helmbold (1961) considered several battles fought over the last two centuries and attempted to formulate Lanchester parameters and characteristics of each, but with mixed results.

Weiss (1966) made an extensive study of the battles of the U.S. Civil War. Some of his results proved quite interesting. Weiss' data indicated a general equality of casualties on both sides with the casualty ratio showing little dependence on the force ratio. Such results tend to indicate that Lanchester's Linear Law may have been applicable in those cases. The results also seem to support beliefs that casualties often tend to remain fairly well balanced until one side begins to lose cohesion.

1.5. Outline of Thesis

The above survey of the literature briefly describes the considerable body of research dealing with Lanchester-type attrition models. Much of this research, however, seems fragmentary. Although an abundance of attrition models has been created, a large percentage are deterministic. Stochastic models which are more realistic are easily described, but have almost always given rise to mathematical difficulties which limit their practical use.

There seems to have been little attempt to proceed beyond the modelling stage, with few efforts at making a decision theoretic analysis of combat situations. The bulk of the work on optimal control (notably that done by Taylor) uses deterministic models and restricts attention largely to problems of allocating fire among various possible classes of targets. Assignments of costs to the employment and destruction of friendly units and rewards for success are virtually non-existent. Thus the important strategic and tactical question of when "victory" becomes too expensive to warrant the expenditure of lives, resources and time necessary to achieve it is all but ignored.

In an attempt to approach some of these problems, this thesis formulates certain decision problems based on an idealized combat environment, and presents some stochastic models and methodology useful in the analysis of such problems. The most promising analytic techniques involve approximations based on martingale central limit theorems and diffusion processes. A

numerical analysis of some hypothetical combat situations is presented, based on the approximations developed. The theoretical approximation methods are shown to yield accurate results and thus provide an important approach to the study of stochastic Lanchester models and their use in a decision theoretic analysis.

Chapter 2 introduces the mathematical framework of the combat decision problem, and outlines the assumptions generally made in order to model combat as a Markovian process. One approximation, for use with the stochastic Linear Law model, is presented and employed to solve a one-stage decision problem. A table of some numerical results of this method is included.

Chapter 3 introduces the martingale method of approximation for use in the solution of the one stage problem. The theory, in terms of martingale central limit theorems, which underlies this approach is discussed, and the basic technique of how to define a martingale from the combat process is outlined. Some numerical results and normal plots are presented to indicate the accuracy of the approximations. In addition, a table of results obtained by using the martingale approach to the solution of the one stage decision problem is presented and compared to the corresponding results from the method of Chapter 2.

A more detailed analysis of a multi-stage combat decision problem is presented in Chapter 4. The solution of such problems requires estimation of the distribution of the combat process as a function of time. This distribution is approximated through use of diffusion models. Again, numerical results are presented

to assess the quality of the approximation. The diffusion model and the martingale methods of Chapter 3 are then employed to solve a two-stage decision problem, and some tables of results are included.

Chapter 5 discusses some possible extensions of the basic decision problem in somewhat general terms. The introduction of modified battle termination criteria as well as a more detailed discussion of uncertainty and prior probability distributions for the parameters of a combat model are included. Finally, it summarizes the work done and discusses other interesting topics for further research.

Chapter 2

COMBAT DECISION PROBLEMS

2.1. The One Stage Problem

The elements of a one stage combat decision problem hereafter referred to as the basic combat decision problem, are based on a relatively simple military situation. The military decision maker, or commander, must decide whether to accept a combat action and, if so, the amount of force to employ. (Note that in this formulation, the commander is basically presumed to be contemplating an offensive rather than defensive combat posture.) The commander's decision is based on his^{*} perception of various pertinent factors, such as the nature of the combat situation, and the strengths, both physical and moral, of friendly and hostile forces. In addition he must consider the costs associated with the employment and destruction of friendly forces, and the relative rewards for victory, balancing these factors to achieve the most favorable result.

The mathematical formulation of such a decision problem may be constructed along the following lines. First, define an underlying space Θ of possible initial conditions which are fundamental to the particular situation and beyond the control of the commander. Next define a space Ω of possible battle outcomes and a space D of decisions available to the commander.

^{*}Present U.S. law prohibits the employment of women in a combat role. For this reason the commander is presumed to be a man.

A real-valued loss function L is defined on $\Omega \times D$ with $L(\omega, \delta)$ representing the loss to the commander when outcome $\omega \in \Omega$ occurs given that he has made the decision $\delta \in D$.

Furthermore, if $P(\cdot | \delta, \theta)$ is a probability distribution on Ω for each $\delta \in D$ and $\theta \in \Theta$ and $F(\cdot)$ a probability distribution on Θ , then

$$P(\cdot | \delta) = \int_{\Theta} P(\cdot | \delta, \theta) dF(\theta)$$

is a probability distribution on Ω for each $\delta \in D$. We will also write $P(\cdot | \delta)$ as $P_{\delta}(\cdot)$.

The expected loss, or risk, of any decision $\delta \in D$ is defined by

$$\rho(\delta) = \int_{\Omega} L(\omega, \delta) dP_{\delta}(\omega).$$

The commander wishes to choose a decision $\delta^* \in D$ such that $\rho(\delta^*)$ is a minimum.

In order to analyze the combat decision problem as outlined above, it is important to understand the character and role of each of its components.

The space Θ of initial conditions includes those elements of the combat situation beyond the immediate control or influence of the commander. Important factors which might be represented by elements of Θ are the numerical strength of hostile forces and their combat power, quantified by their attrition coefficients, as well as the effects of terrain and weather. (Note, the

question of how the enemy chooses his forces will not be considered here.) It will also be assumed that the dynamics of the combat, represented by the mathematical attrition model used to describe it, is also beyond the commander's control. All of these quantities, and perhaps others of interest as well, are considered fixed but possibly unknown. This uncertainty requires the commander to formulate a prior probability distribution F for the possible states in Θ . This prior distribution will be based on the information available to the commander concerning the unknown quantities and his interpretation of that information, as well as his own opinions about the nature of the enemy and situation. Thus Θ represents the underlying structures of the combat situation which may be possible, and F the commander's uncertainty about that structure. In the remainder of this chapter full information will be assumed; that is, the commander's prior places a probability mass of one on some particular element of Θ . The general case is discussed in more detail in Chapter 5.

In the basic combat decision problem, battles continue until the force level of one side is reduced to zero. Since the elements of the outcome space Ω represent the final state of the combat, or terminal point, they may be expressed in the form $(X, 0)$ or $(0, Y)$ where X or Y is the number of survivors of the victorious force. More general battle termination criteria will be discussed in Chapter 5.

Decisions are restricted to determination of the number of units to commit to the combat; all such units are then committed at the beginning of the battle. The basic problem does not address the impact that possible reinforcement during the course of the combat may have on such a decision. (The latter question will be discussed in Chapter 4.) Thus the basic combat problem is couched in the form of a simple one stage decision problem.

The loss function $L(\omega, \delta)$, and the cost and reward structure associated with it, are constructed in terms of a basic unit of value defined as the cost of the destruction of a single friendly unit. The costs for troop employment and the reward for victory are determined in terms of this unit of value. Furthermore, it will be assumed that partial destruction of the enemy force is of no value: if the friendly force is victorious, any reward for eliminating the enemy units is included in the value of victory; if, on the other hand, the friendly force level is reduced to zero, no value is accrued for casualties which may have been inflicted on the enemy. Under these sorts of assumptions the loss function may be written

$$L(\omega, \delta) = cX_0(\delta) + [X_0(\delta) - X_f(\omega)] - VI(\omega) \quad (2.1.1)$$

where $X_0(\delta)$ is the initial friendly force level chosen by decision $\delta \in D$, $X_f(\omega)$ is the surviving friendly force level specified by outcome $\omega \in \Omega$, and $I(\omega) = 1$ if $X_f(\omega) > 0$ and $I(\omega) = 0$ if $X_f(\omega) = 0$. The constants c and V represent the cost of employing troops and the value of victory respectively.

Note that in this case the value of victory does not depend on the number of friendly survivors as long as there is at least one. Alternative formulations in which the surviving force level plays a more direct role in assessing the value of victory will not be discussed here.

Solution of the decision problem requires a knowledge of the expected value of the friendly force level at the conclusion of the combat and the probability of a friendly victory. These quantities are calculated from the probability distributions mentioned above. The distribution $P(\cdot | \delta, \theta)$ is derived from the stochastic combat model appropriate to the state θ . The usual stochastic models of the Lanchester-type are in the form of bivariate or multivariate Markov Chains and these are the types of models which will be employed in the sequel.

Markov chains are characterized first of all by a state space, the elements of which represent the state or condition of the process at any point in time. In stochastic combat models, the state space is generally a cartesian product of the form $\tilde{N} \times \tilde{N} \dots \times \tilde{N}$ where $\tilde{N} = (1, 2, \dots, N)$ for some sufficiently large integer N . The dimension of the space depends on the number of distinct types of units available on either side, each component representing the number of surviving units of a particular type. In the simple case as it has been outlined above, there is only one type of unit on each side and so the state space, E , is simply $\tilde{N} \times \tilde{N}$. The elements of the state space represent the surviving force level on each side.

The dynamics of the combat process are considered Markovian in nature, that is the process will make a transition from its present state to another state according to probability distributions which depend only on the present state, and are independent of the past or future history of the process. The number of states is at most countable, and so the states may be arranged and numbered sequentially in some rational manner. The movement of the process among these states, as a function of time, is characterized by its transition function, $(P_t)_{t \geq 0}$, a family of stochastic matrices. The elements of these matrices are defined by

$$P_t(i, j) = \Pr(z_t = j | z_0 = i) \quad \forall t \geq 0, \quad i, j \in E,$$

where z_t represents the state of the process at time t . (In the bivariate chain, $z_t = (X_t, Y_t)$.) This expression represents the probability that a transition is made from state i to j in a time period of length t .

The transition function P_t is the solution to the forward equations (see Feller (1968))

$$\dot{\tilde{P}}_t = \tilde{P}_t Q, \quad t \geq 0, \quad \text{where} \quad \dot{\tilde{P}}_t = \frac{d}{dt} \tilde{P}_t.$$

The matrix $Q = \dot{\tilde{P}}_0$ is known as the infinitesimal generator. The rows of the Q matrix sum to zero, and its elements allow the determination of the jump function, $\lambda(\cdot)$, and the transition kernel $P(\cdot, \cdot)$. The (i, j) th element of the Q matrix is given by

$$Q(i,j) = \begin{cases} -\lambda(i) & \text{if } i = j \\ \lambda(i)P(i,j) & \text{if } i \neq j. \end{cases}$$

The jump function $\lambda(\cdot)$ defines the holding time of the process in any state i in the sense that this holding time is exponentially distributed with mean $1/\lambda(i)$, if $\lambda(i) > 0$. If $\lambda(i) = 0$, then i is an absorbing state. The transition kernel $P(\cdot, \cdot)$ defines the one step transitions; $P(i,j)$ is simply the probability that, given a jump is made from state i , that jump is to state j . In the combat model the transition kernel represents the probability that the next casualty is taken by the force represented by X or that represented by Y . The Q matrix thus represents the infinitesimal or instantaneous behavior of the process in the sense that

$$P(z_{t+h} = j | z_t = i) = Q(i,j)h + o(h) \quad \text{as } h \rightarrow 0.$$

The components of the stochastic combat model as described above may be derived from various physical assumptions inherent in each state $\theta \in \Theta$. Several examples of such assumptions and the processes defined by them are given in Karr (1974). For example, the assumptions which generate the usual stochastic form of the Lanchester Linear Law are as follows:

- a. All units on each side are identical.
- b. The time required for an X unit to detect a particular Y unit is exponentially distributed with mean $1/s_1$ where s_1 is some positive constant. Each X unit detects Y units independently.

- c. An X unit attacks every Y unit it detects; the conditional probability of a kill given attack is q_1 . The attack occurs instantaneously and contact with the target is lost immediately. No attack may occur without detection.
- d. Y units satisfy assumptions b and c with parameters s_2 and q_2 .
- e. The detection and attack processes of all units present initially are mutually independent.

The process $\{(X_t, Y_t), t \geq 0\}$ has states of the form (i, j) , $i = 0, 1, 2, \dots, X_0$, $j = 0, 1, 2, \dots, Y_0$. Under the above assumptions $a - e$, the infinitesimal generator is given by

$$\begin{aligned}
 Q((i, j), (i, j-1)) &= k_1 ij \\
 Q((i, j), (i, j)) &= -ij(k_1 + k_2) \\
 Q((i, j), (i-1, j)) &= k_2 ij \\
 Q(\cdot, \cdot) &= 0 \quad \text{otherwise}
 \end{aligned}$$

where

$$k_l = s_l q_l, \quad l = 1, 2.$$

From the elements of Q we find the jump function $\lambda(i, j) = ij(k_1 + k_2)$ and the transition kernel

$$P((i,j),(i,j-1)) = \frac{k_1}{k_1 + k_2}$$

$$P((i,j),(i-1,j)) = \frac{k_2}{k_1 + k_2}$$

$$P((i,j),(l,m)) = 0 \quad \text{otherwise.}$$

The probability distribution $P(\cdot | \delta, \theta)$ referred to in the outline of the decision problem may be derived from the transition function $(P_t)_{t \geq 0}$ appropriate to the initial conditions by

$$P(\omega | \delta, \theta) = P_{\infty}^{\theta}(e(\delta, \theta), \omega)$$

where $\omega \in \Omega \subset E$, $e(\delta, \theta) \in E - \Omega$. The state $e(\delta, \theta)$ may be interpreted as the initial state of the process as determined by $\theta \in \Theta$ and $\delta \in D$. Thus we may derive the probability distributions needed in the decision problem from the appropriate Markov model.

2.2. An Example of the Basic Combat Decision Problem

As an example of the basic combat decision model, consider the stochastic version of the Lanchester Linear Law whose assumptions are outlined above. The process is of the form $\{(X_n, Y_n)\}$ where X_n and Y_n are the force levels after a total of n casualties. If the initial force levels are X_0 and Y_0 respectively, then $X_n + Y_n + n = X_0 + Y_0$.

If we define $\Delta X_n = X_{n+1} - X_n$ and similarly define $\Delta Y_n = Y_{n+1} - Y_n$ then the transition kernel of the process may be expressed in the following form:

$$P[(X_n, Y_n), (X_n - 1, Y_n)] = P[\Delta X_n = -1, \Delta Y_n = 0 | (X_n, Y_n)] = \frac{a}{a + b} = q$$

$$P[(X_n, Y_n), (X_n, Y_n - 1)] = P[\Delta X_n = 0, \Delta Y_n = -1 | (X_n, Y_n)] = \frac{b}{a + b} = p.$$

The above holds for $X_n, Y_n > 0$. The attrition constants a and b are analogous to k_1 and k_2 of the previous section, and $p + q = 1$.

As can be seen, the transition probabilities for the model are state independent provided both sides have survivors. States in which either force level has dropped to zero are absorbing states and no transitions are possible. Thus the elements $\omega \in \Omega$ are of the form $(X, 0)$ or $(0, Y)$ with $X \leq X_0$, $Y \leq Y_0$. In this case the combat process takes on the form of a restricted random walk in the plane, beginning at the point (X_0, Y_0) with steps either to the left or downward. (See Figure 2.2.1.) (Note the similarity to the Banach Matchbox problem, Feller (1968).)

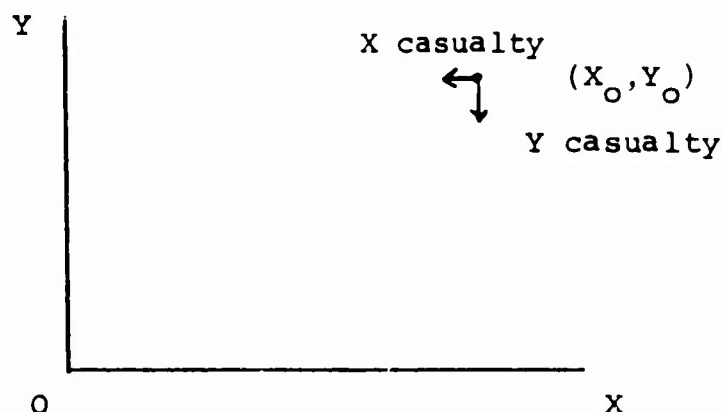


Figure 2.2.1

The X axis and Y axis are absorbing barriers for this walk, and the probability $P(\omega | \delta, \theta)$ may be written in terms of the negative binomial distribution.

Thus, if $\theta \in \Theta$ specifies a linear attrition model of this form, with initial Y force level Y_0 and attrition parameters a and b , and if $\delta \in D$ specifies an initial X force level of X_0 , then the probability of $\omega \in \Omega$, where ω specifies that X wins with X_f survivors is given by:

$$\begin{aligned}
 P(\omega | \delta, \theta) &= P[(X_f, 0) | X_0, Y_0, p] = \binom{X_0 - X_f + Y_0 - 1}{X_0 - X_f} p^{Y_0} q^{X_0 - X_f}, \\
 &\qquad\qquad\qquad 1 \leq X_f \leq X_0 \\
 &\qquad\qquad\qquad (2.2.1) \\
 &= \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c}, \\
 &\qquad\qquad\qquad 0 \leq X_c \leq X_0 - 1
 \end{aligned}$$

where $X_c = X_0 - X_f$ represents the X casualties.

Define the set $\Omega_{X_0} \subset \Omega$ by

$$\Omega_{X_0} = \{\omega \in \Omega \mid \omega = (X, 0), 1 \leq X \leq X_0\},$$

that is the points on the X axis to the left of the initial X force level (excluding the origin). Then the probability of an X victory under the conditions assumed above is

$$P(\Omega_{X_0} \mid X_0, Y_0, p) = \sum_{X_c=0}^{X_0-1} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c}.$$

Thus for a Y_0 and p fixed by $\theta \in \Theta$, the risk of decision $\delta = X_0$ is given by

$$\begin{aligned} \rho(X_0) &= cX_0 + E[X_c \mid X_0, Y_0, p] - VP[\Omega_{X_0} \mid X_0, Y_0, p] \\ &= cX_0 + E[X_c \mid \Omega_{X_0}, X_0, Y_0, p]P[\Omega_{X_0} \mid X_0, Y_0, p] \\ &\quad + E[X_c \mid \Omega_{X_0}^c, X_0, Y_0, p]P[\Omega_{X_0}^c \mid X_0, Y_0, p] - VP[\Omega_{X_0} \mid X_0, Y_0, p]. \end{aligned} \quad (2.2.2)$$

Employing the correct negative binomial expressions in equation (2.2.2) we have

$$\begin{aligned} \rho(X_0) &= cX_0 + \sum_{X_c=0}^{X_0-1} X_c \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c} \\ &\quad + X_0 \left[1 - \sum_{X_c=0}^{X_0-1} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c} \right] \\ &\quad - v \sum_{X_c=0}^{X_0-1} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c} \end{aligned}$$

or

$$\begin{aligned}
 \rho(X_0) = & cX_0 + \sum_{X_c=0}^{X_0-1} X_c \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c} \\
 & + X_0 \sum_{X_c=X_0}^{\infty} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c} \\
 & - v \sum_{X_c=0}^{X_0-1} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c}.
 \end{aligned} \tag{2.2.3}$$

The form of expression (2.2.3) is somewhat awkward, and provides little insight into the qualitative behavior of the risk function. In order to solve the decision problem, the value of X_0 which minimizes (2.2.3) must be obtained. Again, the complexity of expression (2.2.3) renders this task somewhat difficult, requiring an extensive numerical search. Thus the risk function of even this, the simplest of the Lanchester models, presents some serious obstacles to the ready solution of the one-stage decision problem.

In the more complex models, such as that based on the Lanchester Square Law, the mathematical difficulties are only compounded; even basic probabilities such as $P[(X_f, 0) | X_0, Y_0, p]$ involve summations of great complexity (see Smith (1965)). One attempt to circumvent the intractability of expression (2.2.3) is presented in Section 2.4. A more general method of attack suitable for a variety of Lanchester models is presented in Chapter 3. For the moment, however, let us consider a

generalization of the one stage problem which may occur when reinforcements are available to the commander once the battle has begun.

2.3. The Multi-Stage Decision Problem

One possible extension of the one stage problem presented in Section 2.1 is to consider the effect that the availability of reinforcements may have on the choice of an optimum initial force level. Prior to the initiation of the combat, the commander must choose his initial force level; however, he is informed that at some time T , fixed and known in advance, after the battle has begun he may call for and receive reinforcements, if he so desires, at some specified cost. Subject to the same basic structure outlined in Section 2.1, the commander wishes to choose the optimal initial force level to employ, assuming that he will choose reinforcements in an optimal manner when the time comes. It is thus necessary for the commander to solve a two stage decision problem.

The solution to the two stage problem may be found by the methods of backward induction commonly employed in dynamic programming. The reasoning is as follows. For each element θ of the space Θ of initial conditions, the choice of an initial force level X_0 induces a probability distribution $P_{X_0}^{\theta}(X_T, Y_T)$ on the state, (X_T, Y_T) , of the process at time T . For each force level configuration (X_T, Y_T) at time T , the optimal level of reinforcement, $X_R^*(X_T, Y_T)$ may be calculated by solving a simple one stage problem along the same lines as that presented in Section 2.1. This one stage problem is modified by the presence of the X_T units remaining from the original force. It is as if the first X_T units employed in a one stage problem had zero cost. In this case, the loss function for the last stage is

of the form

$$L_1(X_R, \omega) = c_1 X_R + (X_T + X_R - X_f(\omega)) - VI(\omega) \quad (2.3.1)$$

where c_1 is the cost paid for reinforcing units and X_R is the amount of such reinforcements. The quantity $X_T + X_R$ can be considered the "initial" force level for the new one stage problem. Note that the formulation also requires the value of X_R to be non-negative, that is, there is no "credit" given at the final stage if the survivors X_T exceed the optimal force level required by the new one stage problem.

The risk function of the modified one stage problem is obtained by integrating expression (2.3.1) over the space Ω with respect to the probability distribution induced on Ω by the conditions (force levels, attrition structure, etc.) prevalent at time T in a manner exactly analogous to that employed for the basic one stage problem of Section 2.1. The optimal amount of reinforcements for a particular configuration (X_T, Y_T) is that value $X_R^*(X_T, Y_T)$ which satisfies

$$\rho_1(X_R^*(X_T, Y_T)) = \min_{0 \leq X_R < \infty} \rho_1(X_R) \quad (2.3.2)$$

and is, in general, a function of (X_T, Y_T) . For simplicity we write $\rho_1(X_R^*(X_T, Y_T))$ as $\rho_1^*(X_T, Y_T)$.

We may write the risk incurred from making the decision $X_0(\delta)$ as the initial (time 0) force level, arriving at configuration (X_T, Y_T) at time T , and proceeding optimally at that point as

$$\rho_2^T(X_0(\delta)) = cX_0(\delta) + (X_0(\delta) - X_T) + \rho_1^*(X_T, Y_T).$$

The overall risk of choosing an initial force level $X_0(\delta)$ and choosing reinforcements at time T according to the optimal modified one stage procedure outlined above is thus

$$\rho_2(X_0(\delta)) = \int_{\{(X_T, Y_T)\}} \rho_2^T(X_0(\delta)) dP_{X_0}^\theta(X_T, Y_T) \quad (2.3.3)$$

where the integration is performed over the set of all allowable (X_T, Y_T) points.

The commander, therefore, wishes to choose a decision $\delta^* \in D$ such that expression (2.3.3) evaluated at $X_0(\delta^*)$ is minimal. That is

$$\rho_2(X_0(\delta^*)) = \min_{\delta \in D} \rho_2(X_0(\delta)).$$

The extension of the two-stage problem to one with n stages is no more difficult, conceptually, than the extension from one stage to two. Such an extension does, however, give rise to ever increasing computational complexities.

In the n -stage problem, reinforcements are made available at each of n time points, or epochs (T_1, T_2, \dots, T_n) where $T_1 = 0$ and the "reinforcement" available at T_1 represents the initial force level. Again the procedure is based on backward induction. Given the state of the system at epoch T_n , (X_{T_n}, Y_{T_n}) , the optimal level of reinforcements to be introduced at this last stage may be calculated using the modified one stage procedure as introduced in the discussion of the two-stage problem. Similarly, given the

state of the process at epoch T_{n-1} , the optimal level of reinforcements at that stage may be calculated by solving a two-stage problem modified in a manner analogous to that done for the one stage problem.

If we define $\rho_k^*(X_{T_{n-k}}, Y_{T_{n-k}})$ to be the optimal risk attainable when the state of the process prior to the introduction of the k th group of reinforcements is $(X_{T_{n-k}}, Y_{T_{n-k}})$, then

$$\begin{aligned} \rho_k^*(X_{T_{n-k}}, Y_{T_{n-k}}) = E\{c_{n-k}X_{R_k}^* + (X_{T_{n-k}} + X_{R_k}^* - X_{T_{n-k+1}}) \\ + \rho_{k-1}^*(X_{T_{n-k+1}}, Y_{T_{n-k+1}})\} \end{aligned} \quad (2.3.4)$$

where c_{n-k} is the cost of reinforcements at epoch T_{n-k} , and $X_{R_k}^*$ is the optimal level of such reinforcements as a function of $(X_{T_{n-k}}, Y_{T_{n-k}})$. The expectation is taken with respect to the distribution of $(X_{T_{n-k+1}}, Y_{T_{n-k+1}})$ given $(X_{T_{n-k}}, Y_{T_{n-k}})$ and $X_{R_k}^*$.

Thus, the overall risk of choosing an initial force $X_0(\delta)$ and proceeding to choose reinforcements in the optimal manner at each stage is given by

$$\rho_n(X_0(\delta)) = \int_{(X_{T_2}, Y_{T_2})} [cX_0(\delta) + (X_0(\delta) - X_{T_2}) + \rho_{n-1}^*(X_{T_2}, Y_{T_2})] dP_{X_0}^\theta(X_{T_2}, Y_{T_2})$$

where $X_0(\delta) = X_{T_1}$ and $P_{X_0}^\theta(X_{T_2}, Y_{T_2})$ is interpreted as in the discussion of the two stage problem.

Solution of the multi-stage problem is quite difficult due to the highly conditional nature of the probability distributions of the force levels at reinforcement epochs. In the case of a complicated attrition structure, these difficulties are seriously compounded. An example of a two stage problem and its solution, based on an important new technique of approximation, is presented in Chapter 4. This new technique seems to provide the best hope of addressing multi-stage problems.

2.4. One Approach to the Approximate Solution of the One-Stage Problem

The discussion of the one-stage decision problem as presented in Section 2.2 revealed the difficulty of solving for the decision, $X_0(\delta)$, which minimizes the risk function ρ . Typically, closed form solutions of these optimization problems cannot be found due to the complexity and mathematical intractability of the required expressions. This failure of analytic methods leads to a consideration of techniques of approximation, in order to simplify the expressions with which we must deal, and also of numerical methods which may be employed to solve the problem.

One likely approach is to employ a central limit theorem to approximate more cumbersome probability distributions by the more familiar and well studied normal distribution. Consider the Linear Law example presented in Section 2.2. In particular, consider the term

$$\sum_{X_c=0}^{X_0-1} \binom{X_c + Y_0 - 1}{X_c} p^{Y_0} q^{X_c}$$

in expression (2.2.3). This term represents the probability that the X side is victorious conditional on X_0 , Y_0 and p . This probability is merely the probability that a negative binomial random variable is less than X_0 . The standard central limit theorems apply, allowing approximation of this sum by the appropriate value of the standard normal cumulative distribution

function Φ , under the conditions that X_0 and Y_0 are large and p is not very extreme. In combat models, both of the latter assumptions are generally valid.

In this manner, expression (2.2.3) may be written in an approximate form as

$$\rho(X_0) \approx cX_0 + \frac{qY_0}{p} \Phi \left\{ \frac{pX_0 - qY_0 - 1}{(qY_0)^{1/2}} \right\} + X_0 \left\{ 1 - \Phi \left[\frac{pX_0 - qY_0}{(qY_0)^{1/2}} \right] \right\} - v \Phi \left\{ \frac{pX_0 - qY_0}{(qY_0)^{1/2}} \right\}. \quad (2.4.1)$$

We simplify expression (2.4.1) even further by assuming the values of X_0 , Y_0 and p are such that we may consider the arguments of all three normal distribution functions in the above expression to be the same without serious loss in accuracy. (Note that for this reason the usual continuity correction will be ignored as well.) This leads to the approximation.

$$\rho(X_0) \approx (c + 1)X_0 - \Phi \left[\frac{pX_0 - qY_0}{(qY_0)^{1/2}} \right] \left\{ X_0 + v - \frac{qY_0}{p} \right\}. \quad (2.4.2)$$

The optimal decision, X_0^* , is obtained from expression (2.4.2) by differentiating this risk function as a function of X_0 , setting it equal to zero, and solving for X_0 . If we let $\phi(x)$ be the normal probability density function ($\phi(x) = \Phi'(x)$) then we may write

$$\rho'(X_0) = c + 1 - \Phi\left[\frac{pX_0 - qY_0}{(qY_0)^{1/2}}\right] - \left[\frac{p(X_0 + V) - qY_0}{(qY_0)^{1/2}}\right] \varphi\left[\frac{pX_0 - qY_0}{(qY_0)^{1/2}}\right] \quad (2.4.3)$$

The continued presence of the normal distribution function, Φ , remains something of a problem; however, assuming its argument is sufficiently large, we may approximate it through use of the Mills ratio technique. (See Chung (1974), p. 231, Exercise 4.) This approximation allows the estimation of normal tail probabilities by the ratio of the normal pdf to its argument:

$$\text{for } x > 0, \quad 1 - \Phi(x) \simeq \varphi(x)/x.$$

The use of this approximation thus requires the consideration of two cases. Case 1: $pX_0 - qY_0 > 0$ (the subscript 0 is deleted in the sequel). In this case we approximate

$$1 - \Phi\left[\frac{pX - qY}{(qY)^{1/2}}\right] \simeq \frac{(qY)^{1/2}}{pX - qY} \varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right).$$

Thus

$$\begin{aligned} \rho'(X) &\simeq c + \frac{(qY)^{1/2}}{pX - qY} \varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right) - \left[\frac{pX - qY + pV}{(qY)^{1/2}}\right] \varphi\left[\frac{pX - qY}{(qY)^{1/2}}\right] \\ &= c - \varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right) \left\{ \frac{pX - qY}{(qY)^{1/2}} + \frac{pV}{(qY)^{1/2}} - \frac{(qY)^{1/2}}{pX - qY} \right\}. \end{aligned}$$

Since X and Y are assumed to be large, we consider the term $\frac{(qY)^{1/2}}{pX - qY}$ to be negligible. This gives

$$\rho'(X) \simeq c - \varphi \left[\frac{pX - qY}{(qY)^{1/2}} \right] \left\{ \frac{pX - qY}{(qY)^{1/2}} + \frac{pV}{(qY)^{1/2}} \right\}. \quad (2.4.4)$$

Defining $\frac{pX - qY}{(qY)^{1/2}} = \eta$ and $\frac{pV}{(qY)^{1/2}} = \alpha$ we have

$$\rho'(X) = f(\eta) = c - \varphi(\eta)(\eta + \alpha). \quad (2.4.5)$$

In order to solve for the optimal value of X , we set $f(\eta)$ to zero and solve for η , that is, find those values of η such that

$$\varphi(\eta) = \frac{c}{\eta + \alpha} \quad (2.4.6)$$

where c and α are known constants and η is assumed to be positive by definition.

The left-hand side of equation (2.4.6) is the density of the standard normal. The right-hand side, when considered a function of η , is a hyperbola. The number of solutions to equation (2.4.6) is thus the number of points of intersection of two curves. The intersections depend, in turn, on the relative positions of c/α and $\frac{1}{\sqrt{2\pi}}$ as illustrated in Figure 2.4.1. If $c/\alpha < \frac{1}{\sqrt{2\pi}}$, it is clear that there will be one negative solution to equation (2.4.6). However, since η is defined to be positive this solution is invalid. It can, nowever, be shown that there can be at most one positive solution to (2.4.6) if $c/\alpha < \frac{1}{\sqrt{2\pi}}$.

Suppose $\eta_0 < 0$ is a solution to (2.4.6). We wish to determine under what conditions $\eta_0 + \epsilon$ will also be a solution

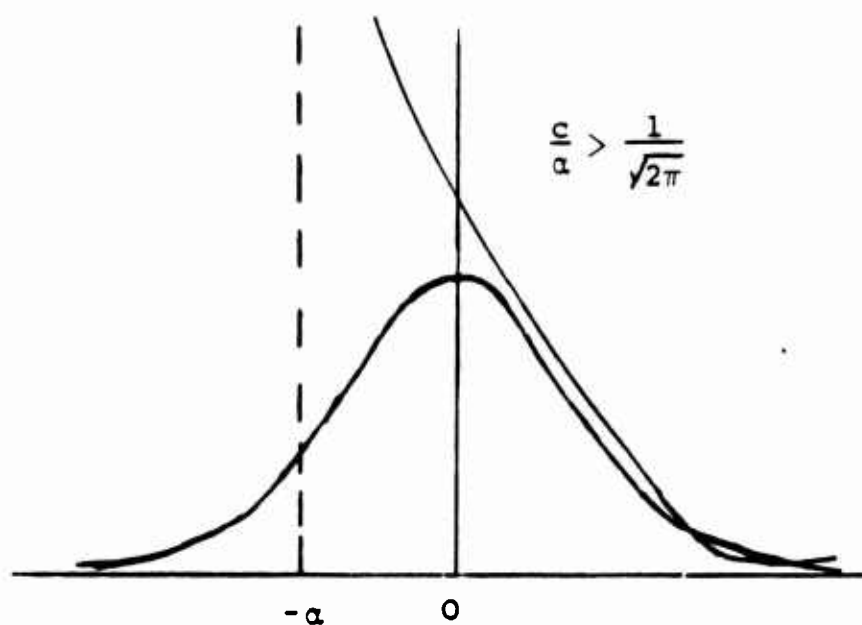
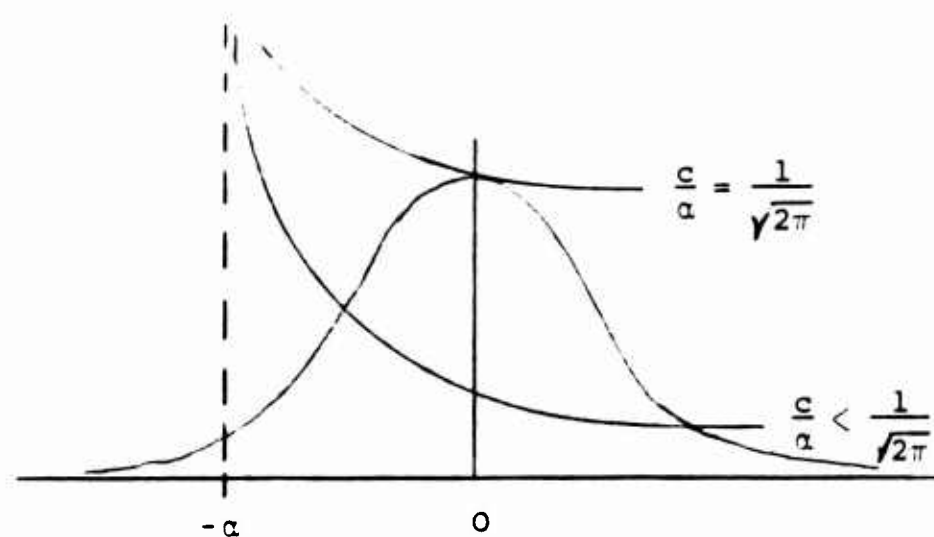


Figure 2.4.1

(where we will assume later that $\eta_0 + \epsilon > 0$) for some $\epsilon > 0$.

We assume $\eta_0 + \epsilon$ satisfies (2.4.6), that is

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} e^{-1/2(\eta_0 + \epsilon)^2} &= \frac{c}{\eta_0 + \epsilon + \alpha} \\ \sqrt{2\pi} e^{1/2(\eta_0 + \epsilon)^2} &= \frac{\eta_0 + \epsilon + \alpha}{c} \\ \sqrt{2\pi} e^{1/2 \eta_0^2} \left[e^{\eta_0 \epsilon + 1/2 \epsilon^2} \right] &= \frac{\eta_0 + \epsilon + \alpha}{c}. \end{aligned} \quad (2.4.7)$$

But since η_0 is a solution, $\frac{1}{\sqrt{2\pi}} e^{-1/2 \eta_0^2} = \frac{c}{\eta_0 + \alpha}$. Thus equation (2.4.7) may be written

$$\begin{aligned} \frac{\eta_0 + \alpha}{c} \left[e^{\eta_0 \epsilon + 1/2 \epsilon^2} \right] &= \frac{\eta_0 + \epsilon + \alpha}{c} \\ e^{\eta_0 \epsilon + 1/2 \epsilon^2} &= \frac{\eta_0 + \epsilon + \alpha}{\eta_0 + \alpha} = 1 + \frac{\epsilon}{\eta_0 + \alpha}. \end{aligned} \quad (2.4.8)$$

Note that when $\epsilon = 0$, both sides of equation (2.4.8) are equal to 1. Also the derivative

$$\left. \frac{d}{d\epsilon} \left[e^{\eta_0 \epsilon + 1/2 \epsilon^2} \right] \right|_{\epsilon=0} = (\eta_0 + \epsilon) e^{\eta_0 \epsilon + 1/2 \epsilon^2} \Big|_{\epsilon=0} = \eta_0 < 0.$$

Thus, if we sketch the functions $e^{\eta_0 \epsilon + 1/2 \epsilon^2} = g_1(\epsilon)$ and

$1 + \frac{\epsilon}{\eta_0 + \alpha} = g_2(\epsilon)$ as functions of ϵ we have Figure 2.4.2.

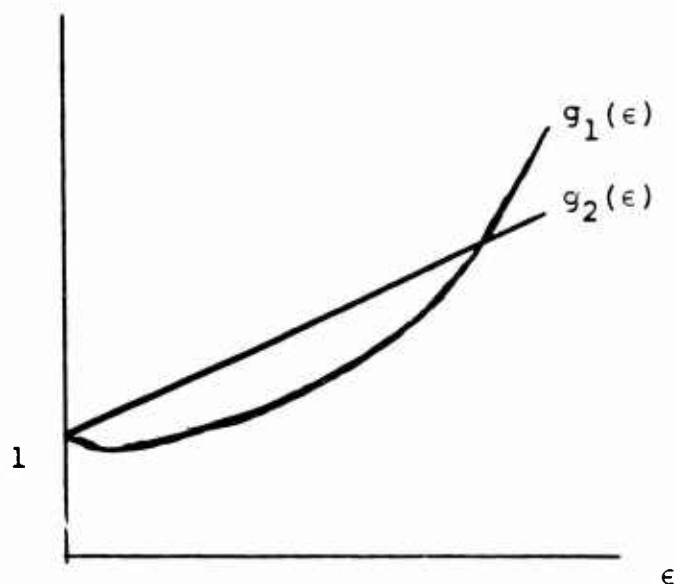


Figure 2.4.2

The slope of the line is $\frac{1}{\eta_0 + \alpha}$ which is assumed to be positive. The slope of the curve is negative at $\epsilon = 0$ but increasing. Thus there can be at most one solution other than η_0 and so at most one positive solution to (2.4.6) when $c/\alpha < \frac{1}{\sqrt{2\pi}}$. In fact, since only positive solutions are admissible in this case ($pX - qY > 0$), there is at most one admissible solution to equation (2.4.6) for $c/\alpha < \frac{1}{\sqrt{2\pi}}$.

Now suppose $c/\alpha > \frac{1}{\sqrt{2\pi}}$. In this case, there can be no negative η solutions to equation (2.4.6). However, the argument employed above is equally valid when the η_0 value employed there is positive. Thus there can be at most two solutions to (2.4.6) when $pX - qY > 0$. A single solution is also possible. Suppose η_0 is the smallest positive solution. Suppose also that $\eta_0 > \frac{1}{\eta_0 + \alpha}$. Then since $e^{\eta_0 \epsilon + 1/2 \epsilon^2}$ is convex, if its

slope at $\epsilon = 0$ is larger than $\frac{1}{\eta_0 + \alpha}$, its graph will always be above the line (see Figure 2.4.3)

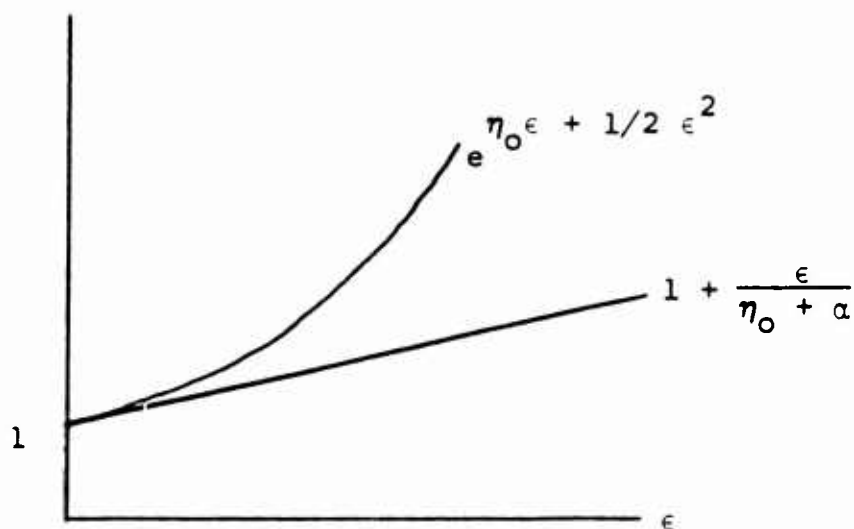


Figure 2.4.3

Thus if the smallest solution η_0 is such that $\eta_0 > \frac{1}{\eta_0 + \alpha}$, it is, in fact, the unique positive solution to (2.4.6).

In summary, if $pX - qY > 0$, there are at most two admissible solutions to equation (2.4.6); that is, at most two critical values for the risk function as a function of the X force level.

Case 2: $pX - qY < 0$. The solution of the problem for this case follows the same lines as that for Case 1. In this manner we arrive at the equation

$$\varphi(\eta) = \frac{c + 1}{\eta + \alpha}. \quad (2.4.9)$$

Where η and α are defined as before but with the restriction that only negative solutions of (2.4.9) are valid. Equation (2.4.9) is of the same form as (2.4.6) and so it follows that there is at

most one valid solution to it, and such a solution can only exist if $\frac{c+1}{\alpha} < \frac{1}{\sqrt{2\pi}}$.

In this latter case, $c/\alpha < \frac{1}{\sqrt{2\pi}}$ (since α is positive). Thus it is possible to have critical values X_1 and X_2 for the risk function such that $pX_1 - qY < 0$ and $pX_2 - qY > 0$. Note also that if $c/\alpha > \frac{1}{\sqrt{2\pi}}$ then $\frac{c+1}{\alpha} > \frac{1}{\sqrt{2\pi}}$ and so there are at most two critical X values for the risk function.

A further examination of the character of the risk function reveals that after a certain point, the risk is a monotonically increasing function of X . There is a point at which the cost of the troops employed is equal to or just greater than the value of victory. As the initial force increases beyond that point, the cost of the large number of troops employed begins to overwhelm the value of victory and the loss and risk increase to infinity as the force level goes to infinity. Thus if there are only two critical points for the risk function, the larger of the two must represent a local minimum or a saddle point.

Furthermore, for very large values of c , it is possible that there are no valid solutions to equations (2.4.6) and (2.4.9). In this case, the risk function has no extreme values except for the boundary value at zero. (That is, the cost of employing troops is so high relative to the value of victory that the combat is best avoided.)

Summarizing the results of the above analysis, the risk function can have zero, one, or two critical values. These findings coincide with intuitive conjectures of possible reasonable

shapes for the risk function based on the proposed loss and reward structure. In general, the risk function can be expected to exhibit one of the qualitative type of behavior exhibited in Figure 2.4.3.

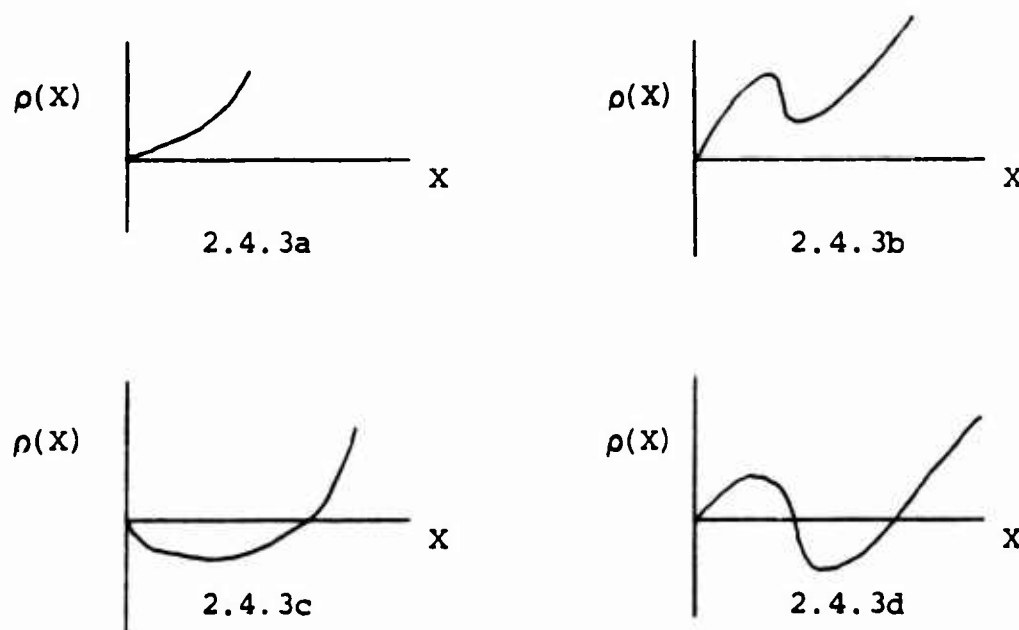


Figure 2.4.3

The difference between functions of the form (2.4.3a) and (2.4.3b) has little impact on decision making. In either of these cases, the optimal value of the risk function occurs for $X = 0$ and no troops should be committed to the combat. In the case of (2.4.3c) and (2.4.3d), however, true optimal force levels exist for positive values.

In order to determine the optimal value of X in these latter cases, it is necessary to solve an equation of the form

(2.4.6) $\frac{1}{\sqrt{2\pi}} e^{-1/2 \eta^2} = \frac{c}{\eta + \alpha}$ for η . It has proven impossible to obtain an analytic solution to such an equation. However, standard numerical techniques such as the bisection method are applicable and proved to give results which are intuitively reasonable and appealing in a minimal amount of computing time. Some specific examples were considered and selected results are included in Table 2.4.1 at the end of the chapter. The optimal force levels are generally rather higher than the minimum force level required for victory by the deterministic model.

Although the basic approach that has been described in this section is relatively straightforward and appears to give reasonable results, it does have some drawbacks. More of a nuisance than a real problem is the fact that the Normal approximation of the probabilities of extreme events (for example, one side suffering only a few casualties) tends to deviate from the actual values to such an extent that large deviation theory must be applied if such events are of great interest. More serious, however, is the fact that this particular approach is not readily applicable to other stochastic Lanchester type models in which the transition probabilities are state dependent. In these cases, the complexity of the expressions for the required probabilities makes an immediate application of standard central limit theorems difficult. One highly effective alternate technique suggested by Watson (1976) will be discussed in Chapter 3.

Table 2.4.1 - Numerical Results for the One Stage Decision Problem
 Lanchester Linear Law, Standard Central Limit
 Theorem Approach

Notation: Y_0 - Initial hostile force level
 p - Probability next casualty is an enemy ($p = \frac{b}{a+b}$)
 c - Cost of employing friendly troops
 V - Reward for destroying entire enemy force
 X_0 - Optimal friendly force level, central limit approach
 ρ_0 - Risk of optimal force
 X_L - Minimal force required to win battle (enemy destroyed with at least one friendly survivor) from deterministic equations.

Y_0	p	c	V	X_0	ρ_0	X_L
100	0.5	0.5	500	136.94	-328.89	101
1000	0.5	0.5	5000	1134.40	-3426.12	1001
100	0.3	0.5	500	299.14	-111.95	234.3
1000	0.3	0.5	5000	2578.38	-1363.33	2334.3
100	0.7	0.5	500	64.92	-423.43	43.86
1000	0.7	0.5	5000	507.58	-4314.08	429.57
100	0.5	0.6	500	135.92	-318.45	101
1000	0.5	0.6	5000	1131.67	-3312.00	1001
100	0.3	0.6	500	296.90	-82.16	234.3
1000	0.3	0.6	5000	2572.47	-1105.31	2334.3
100	0.7	0.6	500	64.40	-416.94	43.86
1000	0.7	0.6	5000	506.16	-4263.16	429.57

Chapter 3

MARTINGALE SOLUTION OF THE ONE STAGE DECISION PROBLEM

3.1. The Watson Martingale Approach

The use of martingales to facilitate the analysis of stochastic systems has proved to be quite an effective technique in many areas of application, and the same holds true in the case of the basic one-stage combat decision problem. The approach presented in this chapter is based on the work of R. K. Watson (1976).

Suppose $\{(X_n, Y_n), n \geq 0\}$ is a discrete stochastic process based on a casualty time scale, that is, X_n and Y_n are the opposing force levels after a total of n casualties have occurred. The usual models for a combat process of this type take the form of bivariate Markov chains. The transition probabilities of such chains may be given in general by

$$(X_{n+1}, Y_{n+1}) = \begin{cases} (X_n - 1, Y_n) & \text{with probability } \frac{g(X_n, Y_n)}{f(X_n, Y_n) + g(X_n, Y_n)} \\ (X_n, Y_n - 1) & \text{with probability } \frac{f(X_n, Y_n)}{f(X_n, Y_n) + g(X_n, Y_n)} \end{cases}$$

for some suitable functions f and g . (See Taylor (1975).)

A discrete time martingale can be defined from this chain by finding a function $K(\cdot, \cdot)$ such that

$$K(X, Y) = [K(X-1, Y)g(X, Y) + K(X, Y-1)f(X, Y)] / [f(X, Y) + g(X, Y)]. \quad (3.1.1)$$

Equation (3.1.1) can, in general, be solved inductively for the function K . Let

$$\frac{g(X_n, Y_n)}{f(X_n, Y_n) + g(X_n, Y_n)} = q(X_n, Y_n) \quad \text{and} \quad \frac{f(X_n, Y_n)}{f(X_n, Y_n) + g(X_n, Y_n)} = p(X_n, Y_n).$$

(That is, $p(X_n, Y_n)$ is the probability that the next casualty is a Y while $q(X_n, Y_n)$ represents the probability that the next casualty is an X .) The function K can then be derived from

$$\begin{aligned} K(X-1, Y) - K(X, Y) &= -p(X, Y) \theta(X, Y) \\ K(X, Y-1) - K(X, Y) &= q(X, Y) \theta(X, Y) \end{aligned} \tag{3.1.2}$$

where θ is some function of (X, Y) which may be chosen in a suitable manner. (See Watson (1976).) Some examples of these types of martingale functions are

$$K_1(X, Y) = pX - qY \quad \text{where} \quad p = \frac{b}{a+b}, \quad q = \frac{a}{a+b}$$

for a Linear Law model with attrition constants a and b , or

$$K_2(X, Y) = \frac{1}{2}[bX(X+1) - aY(Y+1)]$$

for a Square Law model.

Thus, if a function K satisfies equation (3.1.1), the discrete stochastic process $\{K(X_n, Y_n), \mathcal{B}_n, n \geq 0\}$ is a martingale, where $\mathcal{B}_n = \mathcal{B}(X_i, 0 \leq i \leq n)$ is the Borel field generated by $\{X_i, 0 \leq i \leq n\}$. (Note that in the casualty time scale X and Y are functionally related by $X_n + Y_n + n = X_0 + Y_0$. Thus

the σ -field generated by the X, Y pairs is simply that generated by either component.)

The distribution of $K(X_n, Y_n)$ can be approximated using a martingale central limit theorem. Using this approximation, a stochastic analysis of the combat process may be made in a manner somewhat similar to the central limit theorem approach of Section 2.4. Although this idea is due to Watson (1976), his paper does not go into enough detail. It is necessary to explicitly state the martingale central limit theorem employed and to demonstrate that the martingales formed from a combat process satisfy the conditions of that theorem. Difficulties may arise in employing an approximation of this type to the distribution of a stopped martingale, and a triangular array approach seems to be the best way of modelling the situation correctly. These ideas are discussed below.

3.2. Triangular Arrays and Martingale Central Limit Theorems

In order to develop a correct central limit theorem type approximation to the distribution of the martingale $K(X_n, Y_n)$, the problem may best be formulated in terms of a triangular array. Each row of the triangular array will correspond to a different initial (X_0, Y_0) starting point as described below.

Recall that solution of the one-stage decision problem requires knowledge of the probability of victory and the expected number of survivors for the X force. Thus it is necessary to calculate probabilities of the form

$$P(X_f \geq k, Y_f = 0 | X_0, Y_0),$$

where X_f and Y_f are the final force levels of the opposing sides and $0 \leq k \leq X_0$.

Suppose the initial force level configuration, (X_0, Y_0) , is $N \cdot (\delta, \epsilon)$ where N is large and $N\delta$ and $N\epsilon$ are integers greater than zero. The combat process makes transitions from this initial state either to the left or downward in integer steps. (See Figure 3.2.1.) Assume that $N\epsilon \leq N\delta$. After a total of $N\epsilon$ transitions, the force level configuration will lie on the 45° line with equation $X + Y = N\delta$ (see Figure 3.2.1). The point $(N\delta, 0)$ is the first point at which the actual combat process (X_t, Y_t) may be absorbed by its interception of one of the coordinate axes. In general, after $N(\epsilon + \delta) - k$ transitions, the state of the process will be on the line through the points $(0, k)$ and $(k, 0)$.

As seen above, however, for certain values of k it is possible that the actual combat process will be absorbed prior to the $(N(\epsilon + \delta) - k)$ th transition. In order to maintain a consistent approach, it is helpful to define an extension of the actual process to points in the second and fourth quadrants of the cartesian coordinate plane. This extension is defined as follows. If $Y_n = 0$ and $X_n > 0$, then $P(\Delta X_n = 0, \Delta Y_n = -1) = 1$. Similarly, if $X_n = 0$ and $Y_n > 0$, then $P(\Delta X_n = -1, \Delta Y_n = 0) = 1$. That is, once a transition is made onto either axis, all further transitions of the extended process are made in the same direction as the final transition of the actual combat process. Thus the state of the extended process after $N(\epsilon + \delta) - k$ transitions is well defined to lie on the line $X + Y = k$ for $0 \leq k \leq N(\epsilon + \delta)$.

If the function $K(\cdot, \cdot)$ satisfies the condition given in Section 3.1 so that $K(X_n, Y_n) = K_n$ is a martingale, then it becomes necessary to define $K(\cdot, \cdot)$ for (X, Y) points of the extended process in order to maintain the martingale property. This is accomplished by defining

$$K(X, Y) = K(X, 0) \text{ if } Y \leq 0 \text{ and } K(X, Y) = K(0, Y) \text{ if } X \leq 0.$$

This definition of the martingale for points in the extended (X, Y) process preserves the important monotonicity property of the martingale function which is described below.

From equations (3.1.2) it can be seen that the definition of the martingale is such that for either a fixed X value or a fixed Y value the function K is monotonic in the other

variable. Thus, for example, if $\theta(X,0) > 0$, $K(X,0)$ satisfies

$$K(X_2,0) < K(X_1,0) \quad \text{if} \quad X_1 > X_2.$$

In fact, it can be seen that the martingale function K must be monotonic for the sequence of (x,y) points proceeding down a 45° line (see Figure 3.2.2) from upper left to lower right. This property clearly holds for the definition of the extended martingale as proposed above.

This monotonicity property of the martingale function facilitates the easy translation of probabilities of the form

$$P(X_f \geq k, Y_f = 0 | X_0, Y_0) \quad (3.2.1)$$

into terms of martingale exceedance probabilities. The technique is basically as follows (see Figure 3.2.2). For each initial starting configuration (X_0, Y_0) , the extended combat process and corresponding extension of the definition of the martingale function K define as martingale sequence K_0, K_1, \dots, K_{N^*} where N^* is given by $N \cdot (\epsilon + \delta)$ when (X_0, Y_0) is of the form $N(\delta, \epsilon)$. The distribution of K_{N^*} allows calculation of probabilities of the form (3.2.1) due to the monotonicity property (see Figure 3.2.2). For example, if the martingale function $K(x,y)$ is an increasing function of x for a fixed value of y ,

$$P(X_f \geq k, Y_f = 0 | X_0, Y_0) = P(K_{N^*} \geq K_0^* | K_0).$$

Under certain regularity conditions of a type to be discussed below, the distribution of K_{N^*} may be approximated by a normal

distribution for large values of N^* . However, the problem is somewhat more complex than indicated by the preceding argument. Each different initial force level configuration generates its own sequence of martingale values. Therefore, the correct approach to a central limit theorem argument is based on a triangular array; each martingale sequence corresponding to different initial configurations forms a row in a triangular array. Each row in this array is indexed in such a manner that the row index number approaches infinity as the value of N (where $(X_0, Y_0) = N \cdot (\delta, \epsilon)$) approaches infinity. (See Figure 3.2.3.)

Thus, with the problem stated in the form described above, a triangular array martingale central limit theorem, given in Scott (1973) can then be applied. Although Scotts' proof is based on an array with n elements in the n th row, a generalization of the theorem to an array with N_n elements in row n (where $N_n \rightarrow \infty$ as $n \rightarrow \infty$) is immediate. Scotts' full theorem is stated in terms of several sets of equivalent sufficient conditions. A condensed form of this theorem is given as follows.

Theorem 3.2.1. (Scott). Let $\{S_k(n), \mathcal{F}_k(n); 0 \leq k \leq n\}$ be a martingale sequence for all $n \geq 1$ on a probability space (Ω, \mathcal{B}, P) , with $S_0(n) = 0$ a.s., $S_k(n) = \sum_{j=1}^k X_j(n)$ for $1 \leq k \leq n$, $E[S_k^2(n)] = s_k^2(n)$, and $\mathcal{F}_k(n) \supset \mathcal{J}_k(n)$, the σ -field generated by $S_0(n), S_1(n), \dots, S_k(n)$, $0 \leq k \leq n$. Take $s_n^2(n) = 1$ W.L.O.G. Define a sequence of random functions $\eta_n(\cdot)$ on $[0, 1]$ by $\eta_n(t) = S_k(n)$ for $0 \leq t < 1$, $s_k^2(n) \leq t \leq s_{k+1}^2(n)$, and

$0 \leq k \leq n-1$. Let $\eta_n(1) = S_n(n)$. Also define

$$m_n(t) = \max\{m \leq n \mid s_m^2(n) \leq t\} \quad \text{for } 0 \leq t \leq 1, \quad n = 1, 2, \dots$$

Then if

$$\sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{P} t \quad \text{as } n \rightarrow \infty, \quad 0 \leq t \leq 1 \quad (3.2.2)$$

and

$$\sup_{k \leq n} x_k^2(n) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty \quad (3.2.3)$$

then $\eta_n \xrightarrow{D} W$ as $n \rightarrow \infty$ where W is a standard Wiener process on $[0,1]$ and $S_n(n) \xrightarrow{D} X$ as $n \rightarrow \infty$ where X has the standard normal distribution.

If the conditions of Theorem 3.2.1, in a generalized form allowing for N_n elements in the n th sequence, are satisfied, then the distribution of K_N^* can be approximated by a normal distribution for large N (where $N^* = N(\epsilon + \delta)$ and $N \cdot (\delta, \epsilon) = (X_0, Y_0)$).

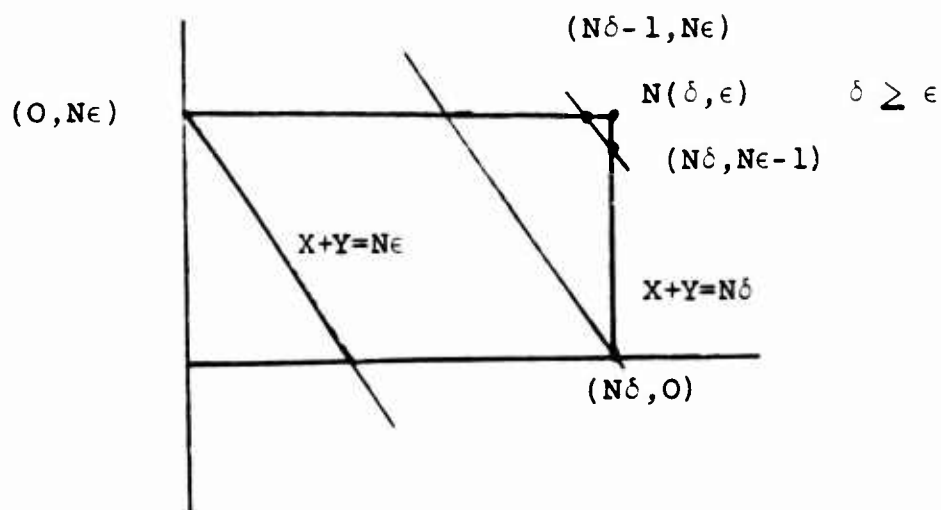


Figure 3.2.1

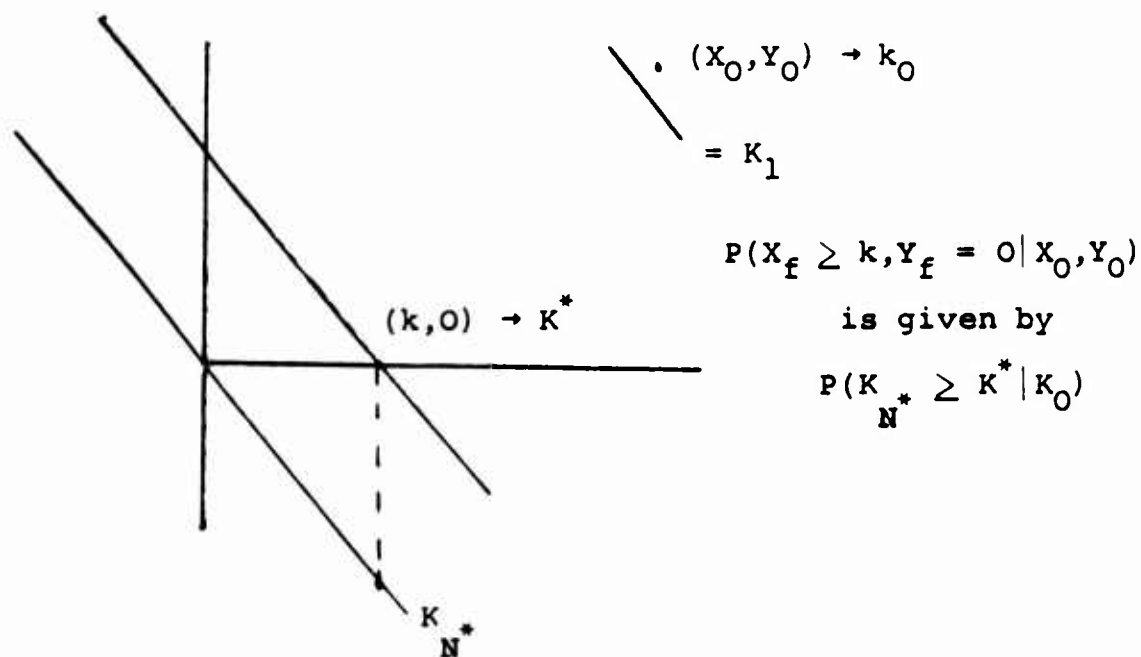


Figure 3.2.2

Figure 3.2.3

$$\begin{array}{lcl}
 K_0(1) & K_1(1) \dots K_{N_1}(1) \\
 K_0(2) & K_1(2) \dots K_{N_2}(2) \\
 \vdots & \vdots \\
 K_0(n) & K_1(n) \dots K_{N_n}(n) \\
 \vdots & \vdots
 \end{array}$$

$$N_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

3.3. An Example

As an example of how the martingale technique can be used, consider the case of the Lanchester Linear Law. As seen in Chapter 2, the linear law model is characterized by constant transition probabilities given by $\Pr(\text{next casualty is a } Y) = \frac{b}{a+b} = p$, $\Pr(\text{next casualty is an } X) = \frac{a}{a+b} = q$ independent of the actual state of the system as long as both the X and Y force levels are positive. Define the function $K(\cdot, \cdot)$ by

$$K(x, y) = bx - ay \quad \text{for } x > 0, y > 0 \quad (3.3.1)$$

and let $K(X_0, Y_0) = \mu$. First it is necessary to show that $K_n = K(X_n, Y_n)$ forms a martingale sequence. Consider points (X_n, Y_n) where $X_n > 0$ and $Y_n > 0$. Then $K(X_n, Y_n) = bX_n - aY_n$.

$$K_{n+1} = K(X_{n+1}, Y_{n+1}) = \begin{cases} K(X_n - 1, Y_n) & \text{w.p. } q \\ K(X_n, Y_n - 1) & \text{w.p. } p \end{cases}$$

so that

$$K_{n+1} = \begin{cases} K_n - b & \text{w.p. } q \\ K_n + a & \text{w.p. } p. \end{cases}$$

Thus

$$E(K_{n+1} | K_n) = \frac{a}{a+b} (K_n - b) + \frac{b}{a+b} (K_n + a) = K_n$$

and the sequence $\langle K_n \rangle$ forms a discrete time martingale with mean μ . (Note that the extension of K to points on the axes and beyond in the manner described previously preserves the martingale property.)

We now wish to show that the martingale K_n , defined above can be used to form a triangular array which will satisfy the conditions of Scott's Theorem 3.2.1.

Let each row in the array be indexed by n , with the n th row to contain N_n elements where $N_n \rightarrow \infty$ as $n \rightarrow \infty$. Define the martingale sequence for the n th row by

$$K_i(n) = \begin{cases} bX_i - aY_i & 0 \leq i \leq N_n, 0 \leq X_i \leq X_0(n), 0 \leq Y_i \leq Y_0(n) \\ bX_i & 0 \leq i \leq N_n, 0 \leq X_i \leq X_0(n), Y_i < 0 \\ -aY_i & 0 \leq i \leq N_n, X_i < 0, 0 \leq Y_i \leq Y_0(n) \end{cases}$$

where $X_0(n)$ and $Y_0(n)$ represent the initial X and Y force levels corresponding to row n . Also define $K_n = K_0(n) = K(X_0(n), Y_0(n)) = bX_0(n) - aY_0(n)$. Finally, let $S_i(n) = K_i(n) - K_0(n)$. Thus

$$\langle S_i(n) \rangle_{i=1}^{N_n}$$

forms a mean zero martingale sequence for all n . Now define

$$m_n(t) = \max\{m \leq N_n \mid s_m^2(n) \leq t\}, \quad 0 \leq t \leq 1, n = 1, 2, \dots$$

where $s_m^2(n) = \text{Var}(S_m(n)) = E(S_m^2(n))$.

We wish to show that the triangular array formed by the $\langle S_i(n) \rangle_{i=1}^{N_n}$ sequences satisfies

$$\sum_{k=1}^{m_n(t)} X_k^2(n) \xrightarrow{P} t \quad \text{as } n \rightarrow \infty, \quad 0 < t \leq 1 \quad (3.2.2')$$

$$\sup_{k \leq N_n} X_k^2(n) \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty, \quad (3.2.3')$$

where (3.2.2') and (3.2.3') are generalizations of expressions (3.2.2) and (3.2.3) allowing N_n elements in row n . These requirements can be shown by demonstrating that

$$s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{p} t \text{ as } n \rightarrow \infty \text{ for } t \in [0,1] \quad (3.3.2)$$

and

$$s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n) \xrightarrow{p} 0 \text{ as } n \rightarrow \infty \quad (3.3.3)$$

are satisfied.

In order to verify (3.3.2) and (3.3.3), it is necessary to determine the order of magnitude of $s_{N_n}^2(n) = \text{Var}[S_{N_n}(n)]$. This is done through the following argument.

Let M_n be the epoch at which absorption of the actual combat process takes place. If $X_0(n)$ and $Y_0(n)$ are the initial force levels, and $X_0(n) \rightarrow \infty$ and $Y_0(n) \rightarrow \infty$ as $n \rightarrow \infty$, then, since $M_n \geq \min(X_0(n), Y_0(n))$, $M_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $K_{M_{n+i}}(n) = K_{M_n}(n)$ for all $i \geq 0$ it follows that

$$\text{Var}[K_{N_n}(n)] = \text{Var}[K_{M_n}(n)].$$

The variance of $K_{M_n}(n)$ may be calculated in the following manner: $E[K_{M_n}(n)] = E[K_0(n)] = \mu_n$.

$$\begin{aligned}
E[K_{M_n}^2(n)] &= E\{E[K_{M_n}^2(n) | K_{M_{n-1}}(n)]\} \\
&= qE\{[K_{M_{n-1}}(n) - b]^2\} + pE\{[K_{M_{n-1}}(n) + a]^2\} \\
&= E[K_{M_{n-1}}^2(n)] + ab.
\end{aligned}$$

And so, iterating the above procedure, it follows that

$$\begin{aligned}
E[K_{M_n}^2(n)] &= \mu_n^2 + M_n ab \quad \text{and so,} \quad \text{Var}[K_{M_n}(n)] = M_n ab, \quad \text{and} \\
\text{Var}[K_{N_n}(n)] &= M_n ab.
\end{aligned}$$

Now, recall that $X_k(n)$ in the expressions (3.3.2) and (3.3.3) is defined by $S_k(n) - S_{k-1}(n)$ and thus can only take on the values a , $-b$, and 0 . Therefore, while $s_{N_n}^2(n) = O(M_n)$, $X_k(n) = O(1)$ for all k , and so (3.3.3) is immediately satisfied. Verification of expression (3.3.2) is a bit more involved. However, it can be shown (Scott (1973)) that (3.3.2) will be satisfied if

$$s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n) \xrightarrow{P} 1 \quad \text{as } n \rightarrow \infty. \quad (3.3.4)$$

Expression (3.3.4) can be verified for the Linear Law case without much difficulty. Since the $X_k(n)$'s are independent (independent transitions for the Linear Law)

$$\text{Var} \sum_{k=1}^{N_n} X_k^2(n) = \sum_{k=1}^{N_n} \text{Var} X_k^2(n) = \sum_{k=1}^{M_n} \text{Var} X_k^2(n)$$

(since $X_k(n) = 0$ for $k > M_n$ by definition). The $\text{Var} X_k^2(n) \leq \max(a^2, b^2) = O(1)$ and so

$$\sum_{k=1}^{M_n} \text{Var} X_k^2(n) = O(M_n).$$

Since $s_{N_n}^4(n) = s_{M_n}^4(n) = O(M_n^2)$,

$$\text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] \xrightarrow{P} 0$$

and by Chebyshev's inequality and the fact that $E\{s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)\} = 1$, it follows that

$$s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n) \xrightarrow{P} 1.$$

Expression (3.3.3) follows from expression (3.3.4) by the same bounding and limiting arguments given by Scott (1973). Thus the martingale defined above for the Lanchester Linear Law model satisfies the conditions (3.3.2) and (3.3.3) of Scott's theorem and so $S_{N_n} \xrightarrow{D} X$, where X has a normal distribution.

The results obtained above allow the approximation of the distribution of $K_{N_n}(n)$ by a normal distribution. The mean of this distribution is given by $\mu_n = bX_0(n) - aY_0(n)$. The variance, however, is not readily calculated. It is possible, however, to approximate the variance of $K_{M_n}(n)$ and thus of $K_{N_n}(n)$ by assuming a large number of transitions prior to absorption of the actual combat process. The technique employed is due to Watson (1976), and is based on a continuous time analog to the discrete time martingale discussed above.

Let $t \in [0, \infty)$ and $K_t = K(X_t, Y_t) = bX_t - aY_t$ with $K_0 = bX_0 - aY_0 = \mu$. Then for the linear law model

$$K_{t+dt} = \begin{cases} K_t - b & \text{w.p.} & aX_tY_tdt + o(dt) \\ K_t + a & \text{w.p.} & bX_tY_tdt + o(dt) \\ K_t & \text{w.p.} & 1 - (a + b)X_tY_tdt + o(dt). \end{cases}$$

The variance of K_{N_n} is approximated by $\text{Var } K_\infty = \lim_{t \rightarrow \infty} \text{Var } K_t = \lim_{t \rightarrow \infty} v(t)$, where $v(t) = \text{var } K_t$. The variance $v(t)$ is, in turn, approximated by $v^*(t) = \text{Var } K_t^*$ where K_t^* is a martingale defined by

$$K_{t+dt}^* = \begin{cases} K_t^* - b & \text{w.p.} & ax_t y_t dt \\ K_t^* + a & \text{w.p.} & bx_t y_t dt \\ K_t^* & \text{w.p.} & 1 - (a + b)x_t y_t dt \end{cases}$$

where x_t and y_t are the deterministic approximations for the stochastic X_t and Y_t process of the K_t martingale. The functions x_t and y_t therefore satisfy the usual Lanchester differential equations.

Since K_t^* is also a martingale, $E[K_t^*] = E[K_{t+dt}^*]$.

$$\begin{aligned} E[K_{t+dt}^{*2}] &= E(E[K_{t+dt}^{*2} | K_t^*]) = ax_t y_t dt (E[K_t^{*2}] - 2bE[K_t^*] + b^2) \\ &\quad + bx_t y_t dt (E[K_t^{*2}] + 2aE[K_t^*] + a^2) \\ &\quad + [1 - (a + b)x_t y_t dt] E[K_t^{*2}]. \end{aligned}$$

Thus

$$\frac{\text{Var}[K_{t+dt}^*] - \text{Var}[K_t^*]}{dt} = ab^2 x_t y_t + a^2 bx_t y_t$$

so

$$v^{*'}(t) = ab^2x_t y_t + a^2bx_t y_t = abx_t y_t(a + b) = dv^*(t)/dt$$

and

$$v^*(t) = \int_0^t dv^*(s) = \int_0^t (a + b)abx_s y_s ds.$$

Now since $ax_s y_s = -dx_s$ and $bx_s y_s = -dy_s$ there are two approaches to the solution of the integral:

$$\begin{aligned} \int_0^t (a + b)abx_s y_s ds &= b(a + b) \int_0^t ax_s y_s ds = b(a + b) \int_0^t -dx_s \\ &= b(a + b)(x_0 - x_t) \end{aligned}$$

or

$$\begin{aligned} \int_0^t (a + b)abx_s y_s ds &= a(a + b) \int_0^t bx_s y_s ds = a(a + b) \int_0^t -dy_s \\ &= a(a + b)(y_0 - y_t). \end{aligned}$$

Thus $v^*(t) = b(a + b)(x_0 - x_t) = a(a + b)(y_0 - y_t)$. (Note that for the linear law, $b(x_0 - x_t) = a(y_0 - y_t)$.)

If x_0 and y_0 are assumed to be large (and therefore x_0 and y_0 as well), the number of transitions which occur prior to absorption is also large, (that is, a very large number of transitions cannot occur in a short period of time) and therefore the time of absorption, t , is considered to be large. The variance of the discrete time martingale at absorption is

approximated by

$$\lim_{t \rightarrow \infty} v(t) \approx \lim_{t \rightarrow \infty} v^*(t).$$

The definition of $v^*(t)$ is such that if $u = bx_0 - ay_0$ is negative, x_∞ is zero while if u is positive, y_∞ is zero.

Thus

$$\begin{aligned} v^*(\infty) = \lim_{t \rightarrow \infty} v^*(t) &= \begin{cases} b(a+b)x_0 & \text{if } u < 0 \\ a(a+b)y_0 & \text{if } u > 0 \end{cases} \\ &= \begin{cases} b(a+b)x_0 & \text{if } bx_0 - ay_0 < 0 \\ a(a+b)y_0 & \text{if } bx_0 - ay_0 > 0. \end{cases} \end{aligned}$$

Thus the distribution of $K_{N_n}(n)$ is approximated by a normal distribution with mean and variance as given above. We use this distribution to approximate $P[X_f \geq k, Y_f = 0] \approx P[n(\mu_n, \sigma_{N_n}^2) > k^*]$ where $k^* = K(k, 0)$ and $\sigma_{N_n}^2$ is the approximate variance of $K_{N_n}(n)$. Both μ_n and $\sigma_{N_n}^2$ depend only on the initial force levels and the attrition parameters of the model. It is thus possible to employ this normal approximation to solve the one-stage decision problem, and also to provide some insight into the combat process without resorting to large scale numerical computation of probabilities.

3.4. Some Numerical Results for the Linear Law Martingale

Several numerical studies were done to assess the accuracy of the normal approximation to the distribution of the K_{N_n} martingale. Results for two selected cases are presented in this section. The technique employed calculated the probability of each possible terminal point on the X and Y axes, given an initial starting configuration and fixed transition probabilities, through iterative procedure. The value of $K(\cdot, \cdot)$ was calculated at each of these points and the induced distribution of K_{N_n} determined. The percentiles of this distribution were then plotted against the normal percentiles. The results are presented in Figures 3.4.1 and 3.4.2.

Figure 3.4.1 represents a case in which the probability of either side taking the next casualty is 0.5. The initial starting values are moderate (100 and 125) with the X side having a numerical superiority. The normal plot is made from the 1st to the 97th percentile and shows quite good agreement with the straight line graph to be expected from a normal distribution.

Figure 3.4.2 represents a somewhat more unbalanced case in which the initial numerical superiority of the X side is more than balanced by the superior effectiveness (higher attrition coefficient α) of the Y side. Once again, the linear nature of the plot indicates that the normal approximation is a reasonable one for initial forces on the order of 10^2 . Although the approximation seems to remain fairly good for somewhat smaller forces, it is best to require the higher order for best results.

Figure 3.4.1

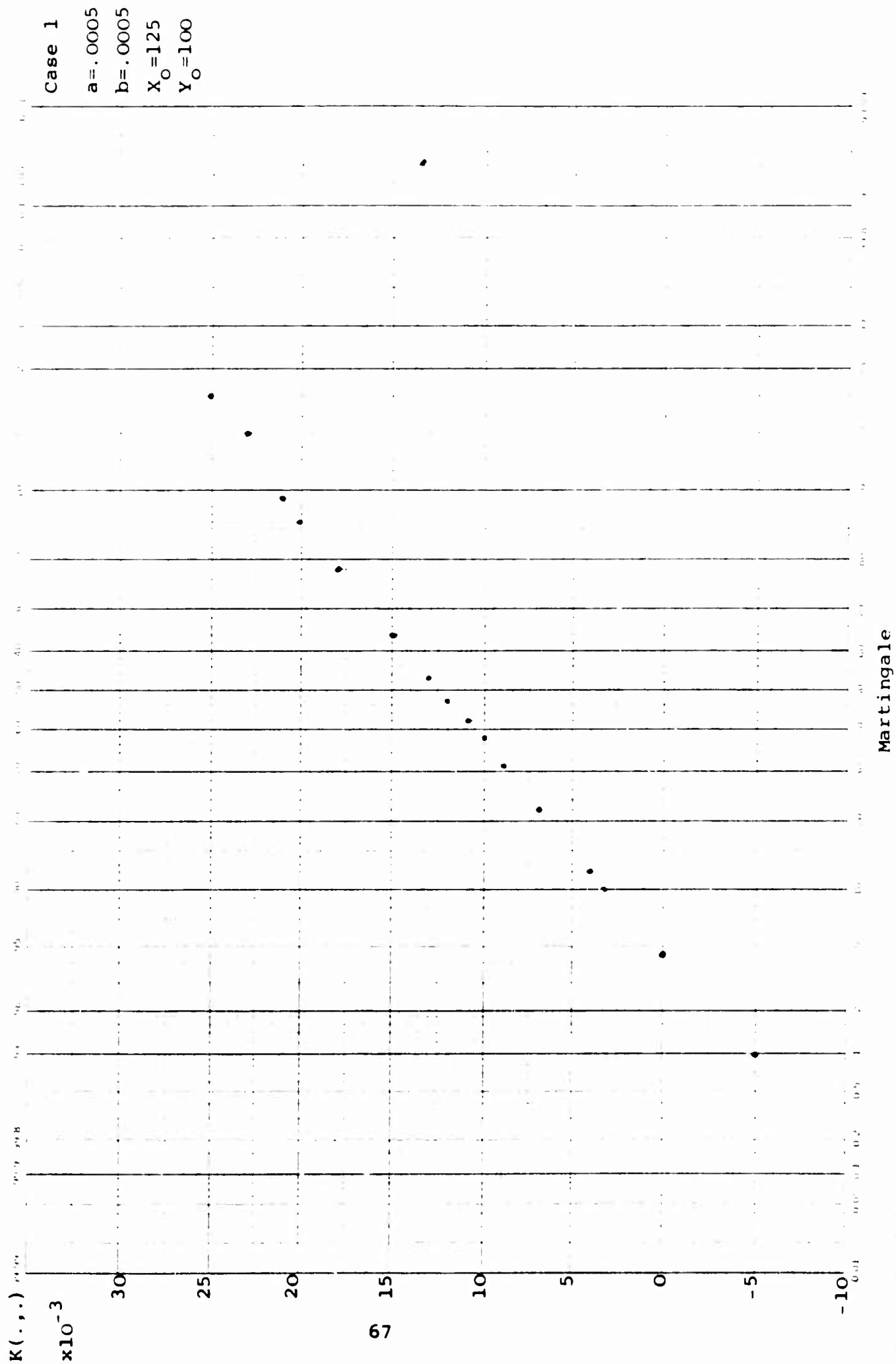
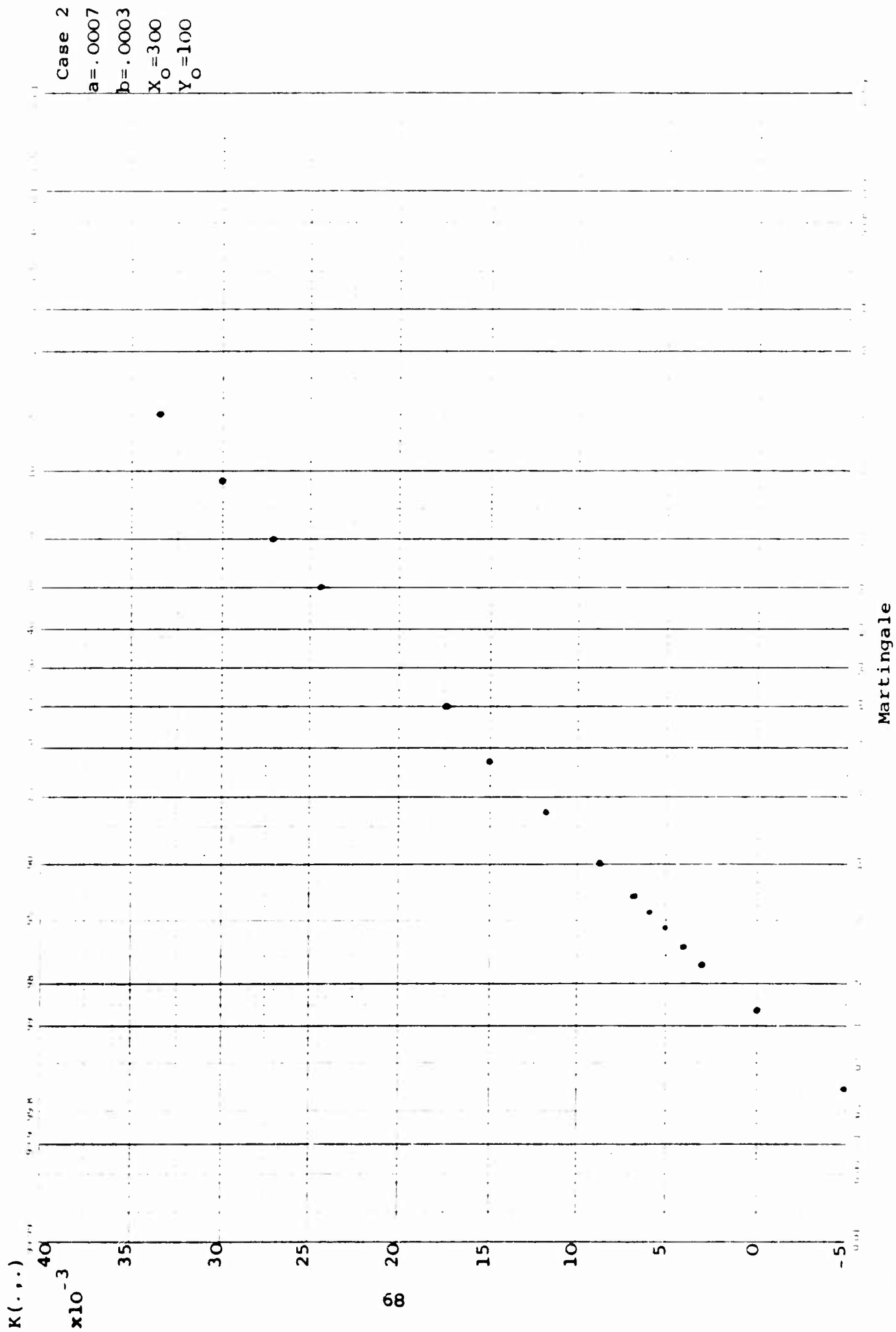


Figure 3.4.2



With the encouraging results from this numerical study completed, it remains to make use of the martingale methods to solve a one stage decision problem. The implementation of the method and some selected results are included in the next section.

3.5. Martingale Solution of the One Stage Decision Problem for the Linear Law Model

The martingale techniques of the previous sections may be used to obtain an approximate solution to a one stage decision problem. Once again, the stochastic model employed will be of the Linear Law type.

Let the initial force configuration be (X_0, Y_0) for X_0, Y_0 large. Define $K(x, y) = px - qy$ where $p = \frac{b}{a+b}$ and $q = \frac{a}{a+b}$. Let $\mu = pX_0 - qY_0$. The distribution of $K_{X_0+Y_0} = K_f$ is approximated by a normal distribution with mean μ and variance σ^2 given by

$$\sigma^2 = \begin{cases} pX_0 & \text{if } \mu < 0 \\ qY_0 & \text{if } \mu > 0. \end{cases}$$

Since the value of K_f is positive if and only if $X_f > 0$ and $Y_f = 0$, $P(X \text{ win}) = P(K_f > 0) \approx \Phi(\mu/\sigma)$ where $\Phi(x)$ is the standard normal distribution function at the point x . The expected value of X_f is obtained by the same type of argument. If $K_f > 0$, then $Y_f = 0$ and $X_f = K_f/p$. Thus

$$E[X_f | X \text{ wins}] = \frac{1}{p} E(K_f | K_f > 0) = \frac{1}{p} [\mu + \sigma \varphi(\mu/\sigma) / \Phi(\mu/\sigma)],$$

where φ is the standard normal density function. So

$$\begin{aligned}
E[X_0 - X_f | X \text{ wins}] &= E[X_c | X \text{ wins}] = X_0 - \frac{1}{p} [\mu + \sigma \varphi(\mu/\sigma) / \Phi(\mu/\sigma)] \\
&= X_0 - \frac{pX_0}{p} + q \frac{Y_0}{p} \frac{\sigma}{p} \frac{\varphi(\mu/\sigma)}{\Phi(\mu/\sigma)} = \frac{q}{p} Y_0 - \frac{\sigma}{p} [\varphi(\mu/\sigma) / \Phi(\mu/\sigma)].
\end{aligned}$$

The risk function, $\rho(X_0)$, then takes on the approximate form

$$\rho(X_0) \approx (c + 1)X_0 - \Phi(\mu/\sigma) [V + X_0 - \frac{q}{p} Y_0] - \frac{\sigma}{p} \varphi(\mu/\sigma). \quad (3.5.1)$$

The solution of the one stage problem requires the minimization of (3.5.1) as a function of X_0 .

It seems clear from the discussion of the shape of the one stage risk function in Section 2.4, as well as from intuitive considerations, that the optimal value of X_0 must lie to the right of the point $\frac{q}{p} Y_0$. In this case $\mu = (pX_0 - qY_0) > 0$ and so $\sigma^2 = qY_0$. Replacing these values in equation (3.5.1) we have

$$\begin{aligned}
\rho(X_0) \approx (c + 1)X_0 - \Phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) [V + X_0 - \frac{q}{p} Y_0] \\
- \frac{\sqrt{qY_0}}{p} \varphi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right). \quad (3.5.2)
\end{aligned}$$

Under the same sorts of assumptions employed in Section 2.4, this risk function may be differentiated, and the optimal X_0 value obtained numerically. This approach was employed for the same cases examined in light of the standard central limit theorem approach of Section 2.4. The results are presented in Table 3.5.1. Table 3.5.2 presents a comparison of the results obtained from these two methods. As can be seen, the agreement of the methods, both in terms of optimal force level and optimal risk, is quite good.

Table 3.5.1 - Numerical Results for the One Stage Decision Problem
Martingale Method

Notation:

- Y_0 - Initial enemy force level
 p - Probability next casualty is enemy
 c - Cost of employing friendly troops
 V - Reward for victory (totally destroying enemy force)
 X_0 - Optimal initial force level
 $\rho(X_0)$ - Risk of optimal force level

Y	p	c	V	X_0	$\rho(X_0)$
100	0.5	0.5	500	136.55	-329.31
1000	0.5	0.5	5000	1134.03	-3426.19
100	0.3	0.5	500	297.66	-112.67
1000	0.3	0.5	5000	2576.84	-1363.92
100	0.7	0.5	500	64.80	-423.49
1000	0.7	0.5	5000	507.46	-4314.12
100	0.5	0.6	500	135.54	-315.71
1000	0.5	0.6	5000	1131.28	-3312.93
100	0.3	0.6	500	295.42	-83.04
1000	0.3	0.6	5000	2570.94	-1106.56
100	0.7	0.6	500	64.28	-417.04
1000	0.7	0.6	5000	506.03	-4263.45

Table 3.5.2 - Numerical Results for the One-Stage Problem

Comparison of Martingale and Standard

Central Limit Theorem

Case	Standard ($X_0, \rho(X_0)$)	Martingale ($X_0, \rho(X_0)$)
1	(136.94, -328.89)	(136.55, -329.31)
2	(1134.40, -3426.12)	(1134.03, -3426.19)
3	(299.14, -111.95)	(297.66, -112.67)
4	(2578.36, -1363.33)	(2576.84, -1363.92)
5	(64.92, -423.43)	(64.80, -423.49)
6	(507.58, -4314.08)	(507.46, -4314.12)
7	(135.92, -318.45)	(135.54, -315.71)
8	(1131.67, -3312.00)	(1131.28, -3312.93)
9	(296.90, -82.16)	(295.42, -83.04)
10	(2572.47, -1105.37)	(2570.94, -1106.56)
11	(64.40, -416.94)	(64.28, -417.04)
12	(506.16, -4263.16)	(506.03, -4263.45)

3.6. Summary

The martingale techniques, based on the work of Watson (1976), have been presented in detail for the case of a Lanchester Linear Law type model. The methods are, however, easily applicable to other stochastic combat models. Watson presents an example of a martingale for the Square Law case. This martingale satisfies the Scott martingale central limit theorem as presented in Section 3.2. (The demonstration of this fact is somewhat extensive and tedious. It has been included in the Appendix.) The martingale technique thus provides a unified and complete solution, at least approximately, to the one stage combat decision problem. The multi-stage problem will be considered in Chapter 4.

Chapter 4

THE SEQUENTIAL COMBAT DECISION PROBLEM AND DIFFUSION MODELS

4.1. Force Level Distributions in Time

The martingale methods presented in Chapter 3 provide useful approximations for the force level distributions upon termination of a battle (that is, when one side is completely eliminated). These methods allow the calculation of approximate solutions to the simple one stage decision problem. Unfortunately, such techniques are not sufficient to solve the multi-stage decision problems presented in Section 2.3. Solution of the latter type of problem requires a knowledge of the distributions of force levels as explicit functions of continuous time.

Time dependent force level distributions for the vast majority of Lanchester type stochastic attrition models are unknown. One expression which does exist is that developed by Clark (1968) for a stochastic version of the Lanchester Linear Law:

Let

$$P(t, m, n) = \text{Prob}\{X(t) = m, Y(t) = n | X(0) = m_0, Y(0) = n_0\},$$

then for $m, n > 0$:

$$P(t, m, n) = \sum_{j=m}^{m_0} \sum_{k=n}^{n_0} \frac{(-1)^{k-n+j-m} a^{m_0-m} b^{n_0-n} (m_0)! (n_0)!}{(a+b)^{n_0-n+m_0-m} m! n! (k-n)! (j-m)! (m_0-j)! (n_0-k)!}$$

$$\cdot \left\{ \prod_{l=1}^{j-m} \frac{(j+n-l)}{(j+k-l)} \prod_{l=1}^{m_0-j} \frac{(n_0-j+l)}{(j+k+l)} \right\} \exp[-(a+b) j k t],$$

where a and b are the usual Lanchester attrition coefficients. Clearly, such an expression is difficult, at best, to work with, especially in the solution of a multi-stage decision problem. Once again, a technique for approximating the actual probability distribution is necessary.

4.2. The Diffusion Approximation. Background

A promising new approach to the approximation of the distribution of force levels as a function of time is based on the construction of a diffusion approximation to the actual combat process. The utility and accuracy of such approximations has been demonstrated for a variety of stochastic processes (for example, see Gaver and Lehoczky (1975, 1976)). In addition, the diffusion technique provides a well studied, unified, and coherent approach to stochastic Lanchester models from which the deterministic equations arise naturally as first order approximations or means of the stochastic processes.

The rationale behind the diffusion approximation is based on the following type of argument. Assume X_0 and Y_0 are large, i.e. $Y_0 = N$, $X_0 = kN$ for some constant k . The combat process can now be studied as $N \rightarrow \infty$.

Let X_t and Y_t be the force levels at time $t \geq 0$. Assume that $\{(X_t, Y_t), t \geq 0\}$ is a time homogeneous Markov process with transition probabilities given by

$$\begin{aligned} P\{(dX_t, dY_t) = (-1, 0) | (X_t, Y_t)\} &= f(X_t, Y_t)dt + o(dt) \\ P\{(dX_t, dY_t) = (0, -1) | (X_t, Y_t)\} &= g(X_t, Y_t)dt + o(dt) \\ P\{(dX_t, dY_t) = (0, 0) | (X_t, Y_t)\} &= 1 - [f(X_t, Y_t) + g(X_t, Y_t)]dt + o(dt) \end{aligned} \quad (4.2.1)$$

where $dX_t = X_{t+dt} - X_t$ and $dY_t = Y_{t+dt} - Y_t$. Further, assume that f and g are smooth, positive functions and that they are of order at least N when X_t and Y_t are of order N (that is, if $X_t = k_1 N$ and $Y_t = k_2 N$ for some positive k_1 and k_2 ,

$f(X_t, Y_t) \geq k_3 N^\alpha$ for some $\alpha \geq 1$, and $k_3 > 0$.) Examples of such functions are:

$f(X, Y) = aXY$, $g(X, Y) = bXY$ as in the Linear Law case, and
 $f(X, Y) = aY$, $g(X, Y) = bX$ as in the Square Law case.

Under the above conditions, the holding time in any state is of order $1/N$ and thus a short interval of time will see the occurrence of a large number of transitions. As can be seen from equations (4.2.1), these transitions are of a Bernoulli nature (that is, a change of either zero or one unit). The total change over any time period is therefore a sum of many Bernoulli-type random variables. The distribution of this total change may then be approximated by a normal distribution according to the central limit theorem. The moments of the approximating normal distribution are derived from the corresponding moments of the Bernoulli distribution. The means of dX_t and dY_t may be easily calculated, based on equations (4.2.1). Upon elimination of terms of order $(dt)^2$, the means are given by $-f(X_t, Y_t)dt$ and $-g(X_t, Y_t)dt$ respectively. In a similar manner it can be shown that the variances are $f(X_t, Y_t)dt$ and $g(X_t, Y_t)dt$.

The distributions of dX_t and dY_t can be treated as approximately normal with the means and variances given above. This naturally suggests that dX_t and dY_t may be modeled in terms of the increments of a Wiener process. If $\{W_t, t \geq 0\}$ is a standard Wiener process, then $dW_t = W_{t+dt} - W_t$ has a $N(0, dt)$ distribution. We can then describe the evolution of the

$\{(X_t, Y_t), t \geq 0\}$ process in terms of two independent Wiener processes W_1 and W_2 by

$$\begin{aligned} dX_t &= -f(X_t, Y_t)dt + \sqrt{f(X_t, Y_t)}dW_1(t) \\ dY_t &= -g(X_t, Y_t)dt + \sqrt{g(X_t, Y_t)}dW_2(t). \end{aligned} \quad (4.2.2)$$

The system (4.2.2) is a stochastic differential equation of the Ito type. (See Arnold (1973) or Gihman and Skorokhod (1972) for a description of these types of equations.)

We can now expand X_t and Y_t in powers of N in a manner similar to that employed in central limit theorems. Specifically we let

$$(X_t, Y_t) = N(x_t, y_t) + \sqrt{N} (Z_1(t), Z_2(t)) \quad (4.2.3)$$

for some large N , where it can be shown (see Kurtz (1971) and Barbour (1974)) that x_t and y_t are deterministic functions satisfying the usual Lanchester-type equations

$$\dot{x}_t = -f(x_t, y_t), \quad \dot{y}_t = -g(x_t, y_t). \quad (4.2.4)$$

The system $\{(Z_1(t), Z_2(t)), t \geq 0\}$ is a stochastic process. By the use of Ito's lemma (Arnold (1973, p. 90) or Gihman and Skorokhod (1972, p. 24) it can be shown that this process satisfies the stochastic differential equation

$$\begin{aligned} dZ_1(t) &= -f(Z_1(t), Z_2(t)) + \sqrt{f(x_t, y_t)} dW_1(t) \\ dZ_2(t) &= -g(Z_1(t), Z_2(t)) + \sqrt{g(x_t, y_t)} dW_2(t), \end{aligned} \quad (4.2.5)$$

where $dZ_i(t) = Z_i(t + dt) - Z_i(t)$, $i = 1, 2$.

The term $N(x_t, y_t)$ in equation (4.2.3) is known as the deterministic approximation of (X_t, Y_t) . The process $\sqrt{N} (Z_1(t), Z_2(t))$ of the same equation is a stochastic "noise" process which is superimposed on the deterministic term to give the actual stochastic process (X_t, Y_t) . In this way, the usual Lanchester equations (4.2.4) can be seen to arise from the stochastic model as a first order approximation.

In order for this diffusion model to be useful in an analysis of a combat system, the distribution of $(Z_1(t), Z_2(t))$ must be derived. If the initial values $Z_1(0)$, $Z_2(0)$ are constants or normally distributed random variables, this distribution is bivariate normal and the mean vector and covariance matrix may be calculated based on the actual form of the stochastic differential equation (4.2.5). Examples of diffusion models for the Lanchester Linear and Square Laws are presented in the following section.

4.3. Diffusion Model Examples

Consider first the case of the Lanchester Square Law. In this case $f(x,y) = ay$ and $g(x,y) = bx$. Employing these in equations (4.2.2) we have

$$\begin{aligned} dX_t &= -aY_t dt + \sqrt{aY_t} dW_1(t) \\ dY_t &= -bX_t dt + \sqrt{bX_t} dW_2(t). \end{aligned} \quad (4.3.1)$$

Similarly, equations (4.2.4) become

$$\dot{x}_t = -ay_t, \quad \dot{y}_t = -bx_t \quad (4.3.2)$$

and the stochastic differential equations (4.2.5) become

$$\begin{aligned} dZ_1(t) &= -aZ_2(t) + \sqrt{ay_t} dW_1(t) \\ dZ_2(t) &= -bZ_1(t) + \sqrt{bx_t} dW_2(t). \end{aligned} \quad (4.3.3)$$

Writing the above equations in matrix notation we have

$$d\tilde{Z}(t) = \tilde{A}Z(t)dt + \tilde{B}_t d\tilde{W}(t) \quad (4.3.4)$$

where

$$\tilde{Z}(t) = (Z_1(t), Z_2(t))', \quad d\tilde{Z}(t) = (dZ_1(t), dZ_2(t))',$$

$$d\tilde{W}(t) = (dW_1(t), dW_2(t))',$$

$$\tilde{A} = \begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{B}_t = \begin{pmatrix} \sqrt{ay_t} & 0 \\ 0 & \sqrt{bx_t} \end{pmatrix}.$$

The equation (4.3.4) is a bivariate, nonstationary linear stochastic

differential equation from which the characteristics of the noise process $(Z_1(t), Z_2(t))$ may be derived (see Arnold (1973), Chapter 8).

Assuming $Z_1(0) = Z_2(0) = 0$, then the distribution of $(Z_1(t), Z_2(t))'$ is bivariate normal. The mean vector of the distribution is $\underline{0}$ and its covariance matrix $\underline{\Sigma}_t$, is the unique non-negative definite solution to the equation

$$\dot{\underline{\Sigma}}_t = A\underline{\Sigma}_t + \underline{\Sigma}_t A' + B_t B_t', \quad \underline{\Sigma}_0 = \underline{0} \quad (4.3.5)$$

where $\dot{\underline{\Sigma}}_t = (\dot{\sigma}_{ij}(t))$ and $\dot{\sigma}_{ij}(t) = \frac{d}{dt} \sigma_{ij}(t)$. Since $\sigma_{12}(t) \equiv \sigma_{21}(t)$, equation (4.3.5) may be rewritten as

$$\begin{aligned} \dot{\sigma}_{11}(t) &= -2a\sigma_{12}(t) + ay(t) \\ \dot{\sigma}_{12}(t) &= -b\sigma_{11}(t) - a\sigma_{22}(t) \\ \dot{\sigma}_{22}(t) &= -2b\sigma_{12}(t) + bx(t) \end{aligned}$$

or

$$\dot{\underline{g}}(t) = A_1 \underline{g}(t) + \underline{K}(t), \quad \underline{g}(0) = \underline{0}, \quad (4.3.6)$$

where

$$A_1 = \begin{pmatrix} 0 & -2a & 0 \\ -b & 0 & -a \\ 0 & -2b & 0 \end{pmatrix} \quad \underline{K}(t) = \begin{pmatrix} ay(t) \\ 0 \\ bx(t) \end{pmatrix}$$

and

$$\underline{g}(t) = (\sigma_{11}(t), \sigma_{12}(t), \sigma_{22}(t))'.$$

Equation (4.3.6) may be solved by standard methods due to the fact that A_1 is not a function of t . The solution is

$$\underline{g}(t) = \exp(-\underline{A}_1 t) \int_0^t \exp(-\underline{A}_1 s) \underline{K}(s) ds \quad (4.3.7)$$

where $\exp(\underline{M})$ is defined by the power series

$$\exp(\underline{M}) = \underline{I} + \underline{M} + \frac{\underline{M}^2}{2!} + \frac{\underline{M}^3}{3!} + \dots$$

The Lanchester equations (4.2.4) can be easily solved for the deterministic system (x_t, y_t) , and give

$$\begin{aligned} x_t &= x_0 \cosh td - cy_0 \sinh td \\ y_t &= y_0 \cosh td - c^{-1}x_0 \sinh td \end{aligned} \quad (4.3.8)$$

where $c = \sqrt{a/b}$ and $d = \sqrt{ab}$. These results, combined with those obtained from the analysis of the stochastic term, yield the diffusion approximation

$$(X_t, Y_t) \sim n(N(x_t, y_t), N\underline{\Sigma}_t)$$

for the force level distribution as a function of time.

The development of the diffusion approximation for the Lanchester Linear Law follows the same basic principle, but has a slightly modified form. We write

$$\begin{aligned} dX_t &= \frac{-a}{N} X_t Y_t dt + \sqrt{\frac{a}{N}} X_t Y_t dW_1(t) \\ dY_t &= \frac{-b}{N} X_t Y_t dt + \sqrt{\frac{b}{N}} X_t Y_t dW_2(t). \end{aligned}$$

Note that the attrition terms are rescaled by dividing out by a factor of N . This allows the changes to remain of order N for X and Y of order N and prevents the system from degenerating too rapidly due to changes of order N^2 (that is, casualties occurring at an unreasonably rapid rate). Proceeding along the same lines as before we have the deterministic system

$$\dot{x}_t = -ax_t y_t, \quad \dot{y}_t = -bx_t y_t \quad (4.3.10)$$

and the stochastic equation

$$d\tilde{Z}(t) = \tilde{A}(t)\tilde{Z}(t)dt + \tilde{B}(t)dW(t) \quad (4.3.11)$$

where

$$\tilde{Z}(t) = (Z_1(t), Z_2(t))', \quad d\tilde{Z}(t) = (dZ_1(t), dZ_2(t))'$$

$$\tilde{A}(t) = \tilde{A}_t = \begin{pmatrix} -ay_t & -ax_t \\ -by_t & -bx_t \end{pmatrix}$$

and

$$\tilde{B}(t) = \tilde{B}_t = \begin{pmatrix} \sqrt{ax_t y_t} & 0 \\ 0 & \sqrt{bx_t y_t} \end{pmatrix}.$$

Equation (4.3.11) is once again a linear equation. Once again, therefore, for $\tilde{Z}(0) = \tilde{Q}$, the distribution of $\tilde{Z}(t)$ is bivariate normal with \tilde{Q} mean and covariance matrix $\tilde{\Sigma}_t$, the unique non-negative definite solution to

$$\dot{\Sigma}_t = A_t \Sigma_t + \Sigma_t A_t' + B_t B_t', \quad \Sigma_0 = 0. \quad (4.3.12)$$

Rewriting equation (4.3.12) in terms of variance components

$\sigma_{11}(t)$, $\sigma_{22}(t)$, $\sigma_{12}(t)$, we have

$$\begin{pmatrix} \dot{\sigma}_{11}(t) \\ \dot{\sigma}_{12}(t) \\ \dot{\sigma}_{22}(t) \end{pmatrix} = \begin{pmatrix} -2ay_t & -2ax_t & 0 \\ -by_t & -(ay_t + bx_t) & -ax_t \\ 0 & -2by_t & -2bx_t \end{pmatrix} \begin{pmatrix} \sigma_{11}(t) \\ \sigma_{12}(t) \\ \sigma_{22}(t) \end{pmatrix} + \begin{pmatrix} ax_t y_t \\ 0 \\ bx_t y_t \end{pmatrix}$$

or

$$\dot{\sigma}_t = C_t \sigma_t + D_t, \quad \sigma_0 = 0. \quad (4.3.13)$$

In this case, however σ_{12} is a linear function of σ_{11} and σ_{22} :

$$\begin{aligned} \sigma_{12}(t) = \frac{1}{2} \frac{b}{a} \sigma_{11}(t) + \frac{1}{2} \frac{a}{b} \sigma_{22}(t) + \frac{1}{2}(x_t - x_0) \\ + \frac{1}{2}(y_t - y_0). \end{aligned}$$

Thus we must solve a reduced problem, seeking the solutions $\sigma_{11}(t)$ and $\sigma_{22}(t)$ to

$$\begin{pmatrix} \dot{\sigma}_{11}(t) \\ \dot{\sigma}_{22}(t) \end{pmatrix} = \begin{pmatrix} -(2ay_t + bx_t) & -\frac{a^2}{b} x_t \\ -\frac{b}{a} y_t & -(2bx_t + ay_t) \end{pmatrix} \begin{pmatrix} \sigma_{11}(t) \\ \sigma_{22}(t) \end{pmatrix} + \begin{pmatrix} ax_t(x_0 + y_0) - ax_t^2 \\ by_t(x_0 + y_0) - by_t^2 \end{pmatrix}. \quad (4.3.14)$$

The solution of equation (4.3.14) depends on the solution of the deterministic equations (4.3.10). The solution to these

latter equations depends in turn on the sign of $(ay_0 - bx_0)$.

The solutions to equation (4.3.10) are given by the following.

Case 1: $(ay_0 - bx_0) < 0$

$$x_t = x_0 - \frac{a}{b}(y_0 - y_t), \quad y_t = \frac{(bx_0 - ay_0)y_0 \exp[(ay_0 - bx_0)t]}{bx_0 - ay_0 \exp[(ay_0 - bx_0)t]} .$$

(4.3.15)

Case 2: $(ay_0 - bx_0) > 0$

$$x_t = \frac{(ay_0 - bx_0)x_0 \exp[(bx_0 - ay_0)t]}{ay_0 - bx_0 \exp[(bx_0 - ay_0)t]}$$

$$y_t = y_0 - \frac{a}{b}(x_0 - x_t).$$

In the very special case $ay_0 = bx_0$, we have

$$y_t = \frac{ay_0}{y_0 t + a}, \quad x_t = \frac{a^2 y_0}{b(y_0 t + a)} .$$

The solution of equation (4.3.14) is, unfortunately, not as straightforward as the solution of equation (4.3.6) due to the time dependence of the A_t matrix in (4.3.14). However, simple computing routines exist (see Dahlquist and Bjorck (1974)) which allow numerical solution to such equations by such classical methods as the Runge-Kutta technique.

Section 4.4 presents some numerical results for the diffusion models, comparing the diffusion approximation to results from a computer simulation of the combat system.

4.4. Numerical Results

This section presents the results of some numerical studies made in an attempt to assess the accuracy of the diffusion approximation at various force level sizes and configurations. The theory presented in Section 4.2 indicate that the diffusion approximation should be a good one for "large" force levels due to the fact that a large number of transitions will take place in a time period of length t . This section presents some selected results to serve as an empirical verification of these theoretical expectations and to answer, albeit in a rather heuristic manner, the question of how "large" is large enough. Results are given first for the Linear Law model and then for the Square Law.

The mean vector for both models are obtained by solving the usual Lanchester equation. Calculation of the covariance matrix for the Square Law requires solution of equations (4.3.6), while that of the Linear Law requires solution of equation (4.3.13). These equations cannot be solved in closed form other than the integral form (4.3.7) for the Square Law. However, the existence of standard, packaged routines* allows numerical solutions to be obtained with relative ease. The procedure employed here is based on a modified form of the classical Runge-Kutta method. In general, the truncation error of this fourth-order technique is of order h^5 where h is the interval length employed in the routine. Further discussion of the errors of this method is given in Gill (1951). Various sizes of h were considered

*The actual routine used is subroutine RKDE in the Univac 1108 Math-Pack, with a minor addition. See the documentation for more details.

and the value 0.001 proved to give results equivalent to those obtained from a much finer mesh in the case of the Linear Law and the Square Law.

Standard Monte Carlo techniques were employed to construct a simulation of combats based on both the Linear and Square Law dynamics. The outcomes of these experiments were given in terms of the survivors on either side at some specified time T . Samples of size 2000 were obtained for the Linear Law and 3000 for the Square Law. The sample mean vector and covariance matrix were calculated for each case and compared to the result obtained from the solutions to the diffusion equations. As a further empirical display, the marginal distribution of the X and Y survivors was plotted against the normal percentiles. Plots of this type were also made for a 45° rotation of the coordinate axes. Summaries of the moment comparisons and the normal plots for the cases considered are given on the following pages. These provide empirical evidence of the quality of the normal approximations.

Comments on the results for each model are made at the end of the appropriate tables and figures.

Table 4.4.1a - Comparison of Simulation and Diffusion Approximation

Linear Law Case

Parameter Values: $a = 0.0005$ $b = 0.0005$ $X_0 = 125.0$ $Y_0 = 100.0$ $T = 15.0$

	<u>Simulation</u>	<u>Diffusion</u>
X Mean	74.381	74.233
X Standard Deviation	6.110	6.161

Relative Difference of Standard Deviation

$$\left(\frac{SD \text{ Sim} - SD \text{ Dif}}{SD \text{ Dif}} \right) = -0.008$$

	<u>Simulation</u>	<u>Diffusion</u>
Y Mean	49.112	49.233
Y Standard Deviation	5.404	5.463

Relative Difference of Standard Deviation = -0.012

	<u>Simulation</u>	<u>Diffusion</u>	<u>Rel. Difference</u>
Covariance	-16.363	-16.839	-0.029
Correlation	-0.496	-0.499	-0.009

Table 4.4.1b

Parameter Values: $a = 0.0007$ $b = 0.0003$
 $x_0 = 300.0$ $y_0 = 100.0$
 $T = 15.0$

	<u>Simulation</u>	<u>Diffusion</u>
X Mean	157.568	157.304
X Standard Deviation	11.676	11.497

Relative Difference of Standard Deviation = 0.015

	<u>Simulation</u>	<u>Diffusion</u>
Y Mean	38.943	38.845
Y Standard Deviation	5.555	5.415

Relative Difference of Standard Deviation = 0.025

	<u>Simulation</u>	<u>Diffusion</u>	<u>Rel. Difference</u>
Covariance	-41.386	-39.393	0.048
Correlation	-0.638	-0.633	0.008

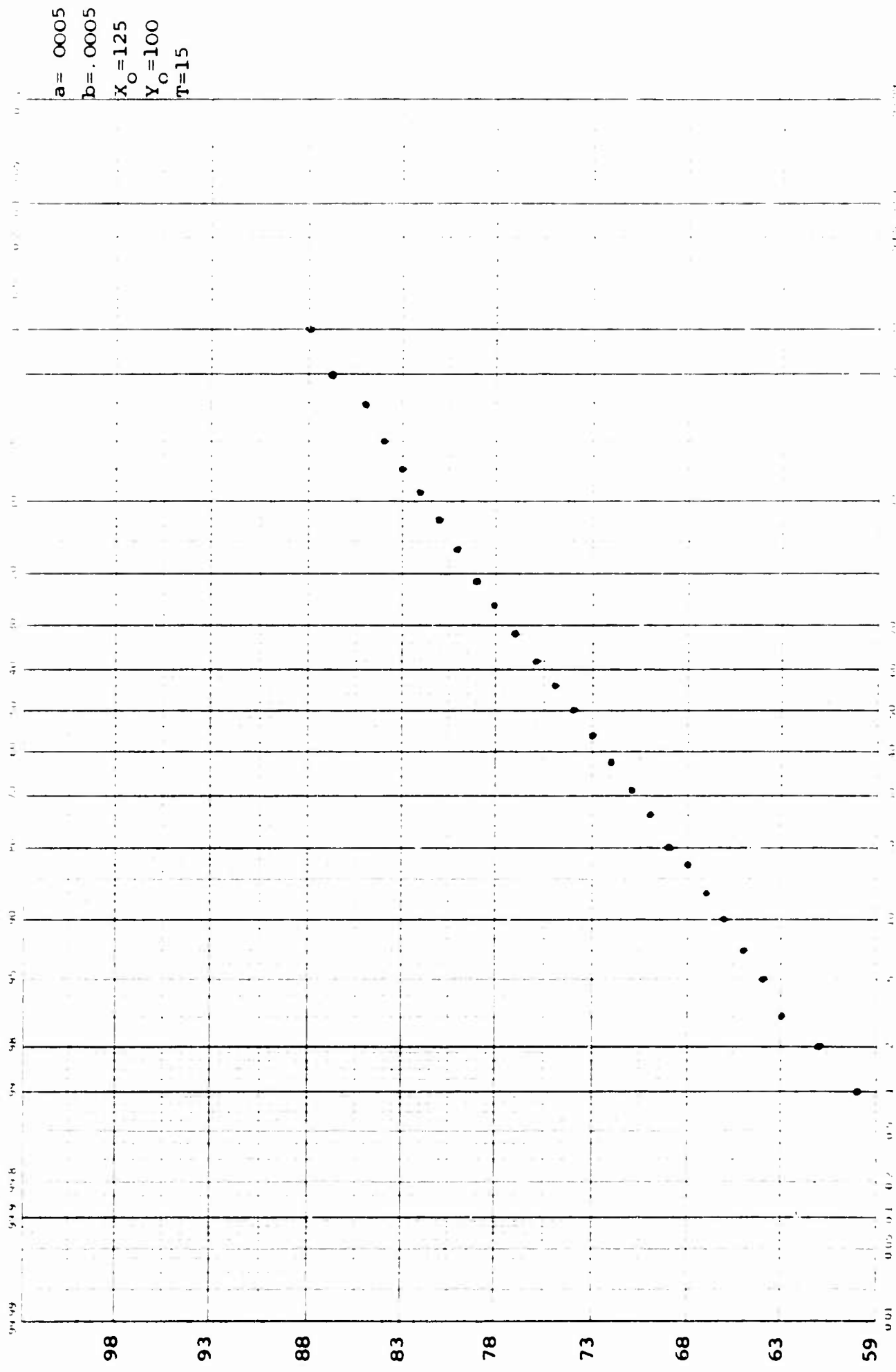


Figure 4.4.1a

X Marginal - Diffusion Linear Law

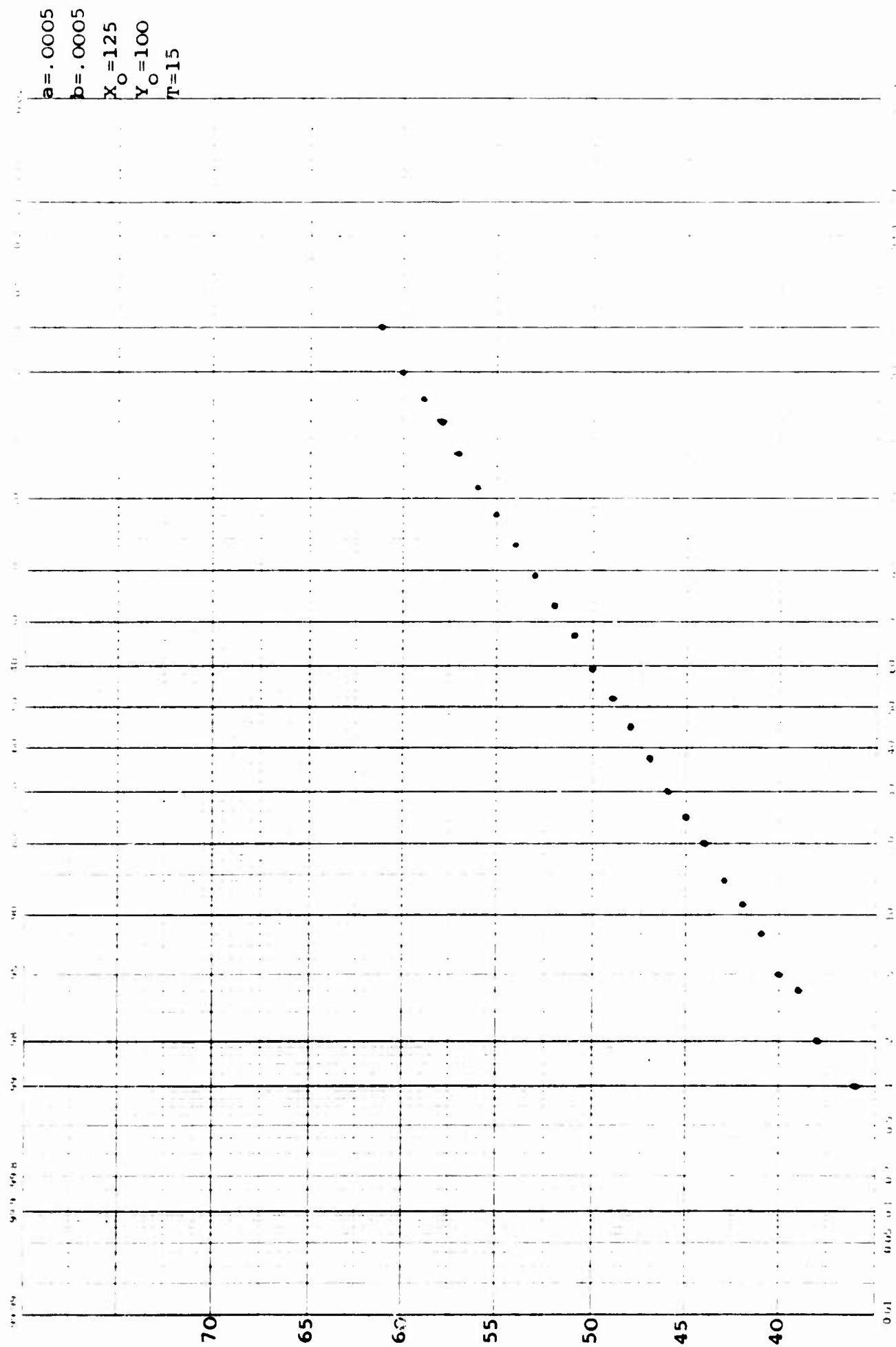
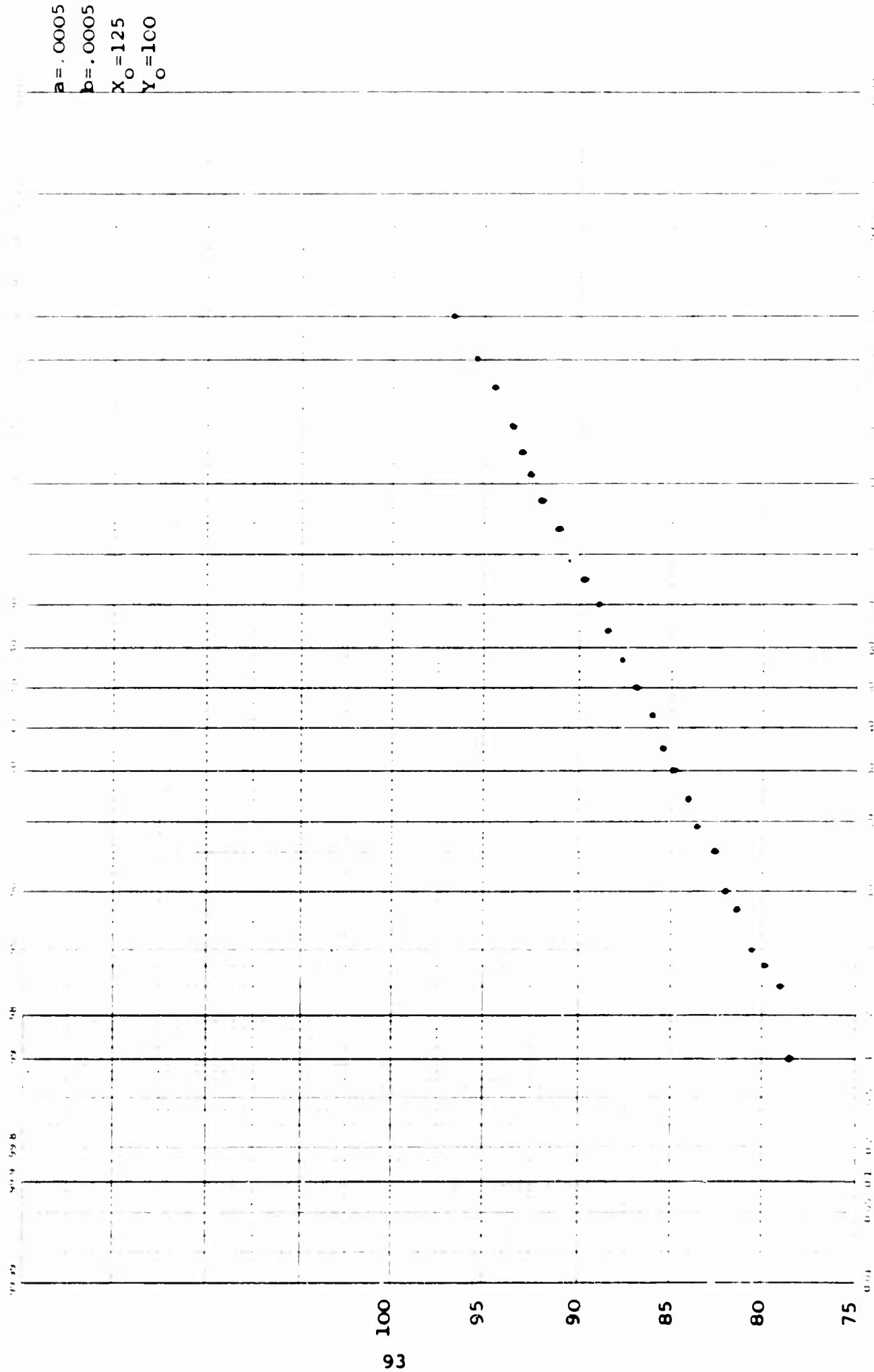
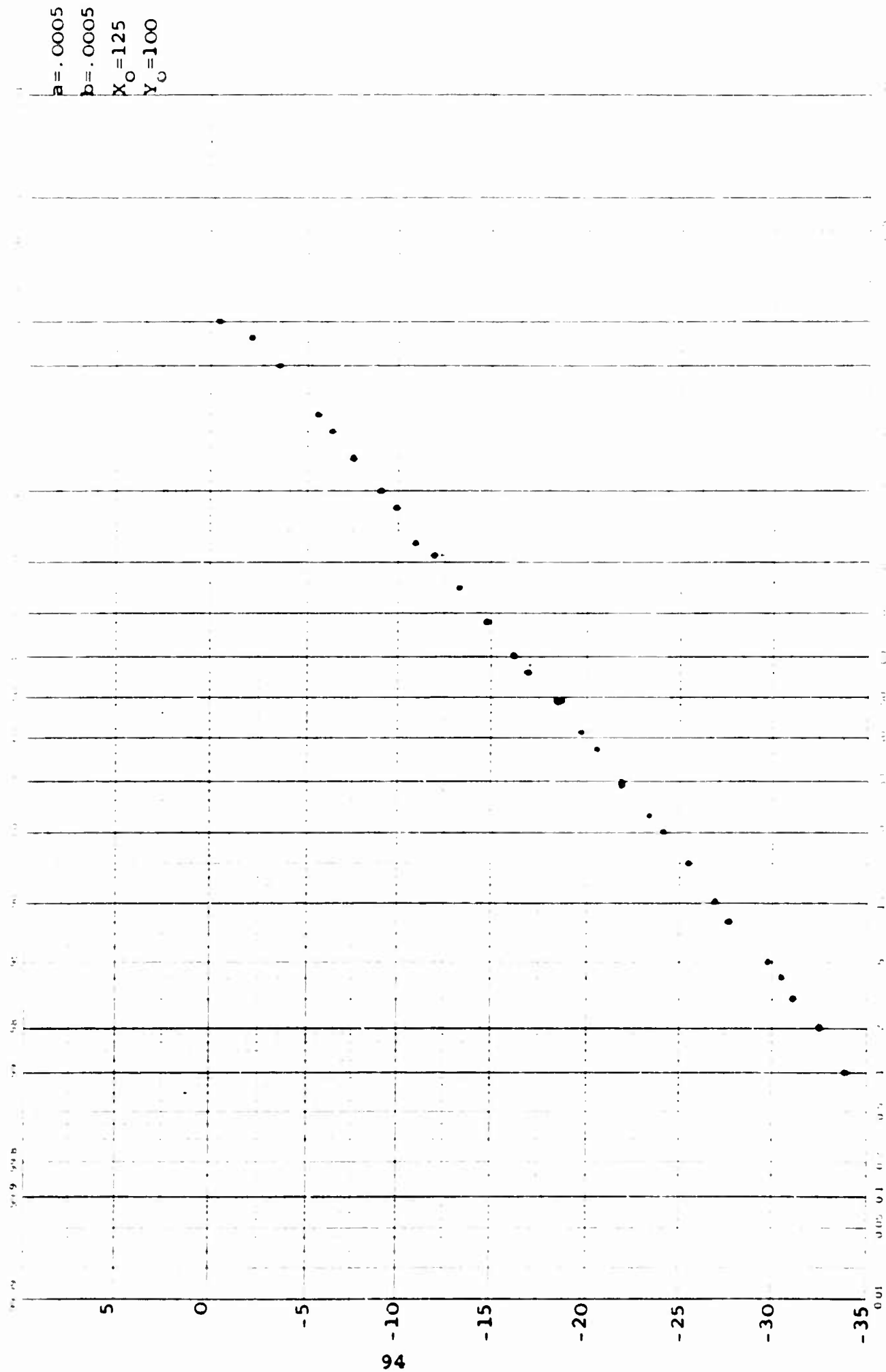


Figure 4.4.1b

Y Marginal - Diffusion Linear Law



Totated X Marginal - Diffusion (45° rotation) Linear Law Figure 4.4.2a



Rotated Y Marginal - Diffusion (45° rotation) Linear Law Figure 4.4.2b

Comments: As can be seen from Tables 4.4.1a and 4.4.1b, the mean vector and covariance matrix calculated from the diffusion model agree quite closely with the corresponding sample moments of the simulation experiments. These results are typical. Also, the plots of the marginal distributions and their rotations show a rather good normal fit for the center 98% of the distributions obtained from the simulation samples. The tails, which are not included in the plot, tended to show somewhat more pronounced deviation from the normal, indicating the advisability of a more careful and refined analysis, perhaps including the use of large deviation theory, if the probability of extreme events is of some importance. Otherwise, the diffusion based normal approximation seems quite adequate for force levels of this size and larger.

Table 4.4.2a - Comparison of Simulation and Diffusion Approximation
Square Law Case

Parameter Values: $a = 0.05$ $b = 0.05$
 $X_0 = 300$ $Y_0 = 300$
 $T = 15.0$

	<u>Simulation</u>	<u>Diffusion</u>
X Mean	141.579	141.710
X Standard Deviation	15.532	15.422

Relative Difference of Standard Deviation

$$\left(\frac{SD \text{ Sim} - SD \text{ Dif}}{SD \text{ Dif}} \right) = 0.007$$

	<u>Simulation</u>	<u>Diffusion</u>
Y Mean	141.860	141.710
Y Standard Deviation	15.163	15.422

Relative Difference of Standard Deviation = -0.017

	<u>Simulation</u>	<u>Diffusion</u>	<u>Rel. Difference</u>
Covariance	-163.746	-163.079	0.004
Correlation	-0.695	-0.686	0.014

Table 4.4.2b

Parameter Values:

 $a = 0.07$ $b = 0.03$ $X_0 = 500.0$ $Y_0 = 200.0$ $T = 10.0$

	<u>Simulation</u>	<u>Diffusion</u>
X Mean	408.366	408.473
X Standard Deviation	11.096	11.135

Relative Difference of Standard Deviation = -0.003

	<u>Simulation</u>	<u>Diffusion</u>
Y Mean	66.231	66.065
Y Standard Deviation	12.235	12.159

Relative Difference of Standard Deviation = 0.006

	<u>Simulation</u>	<u>Diffusion</u>	<u>Rel. Difference</u>
Covariance	-68.784	-69.429	-0.009
Correlation	-0.507	-0.513	-0.012

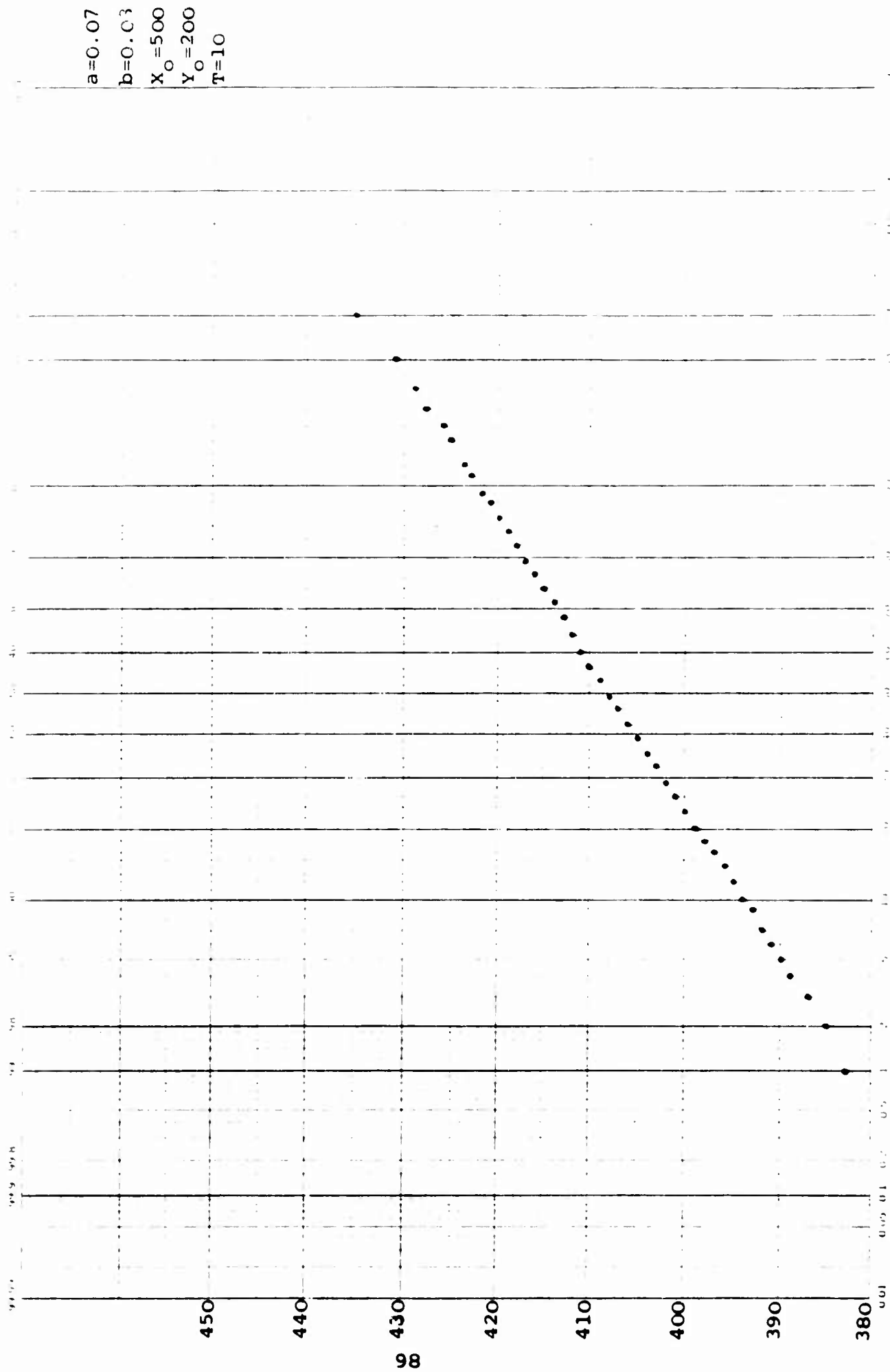
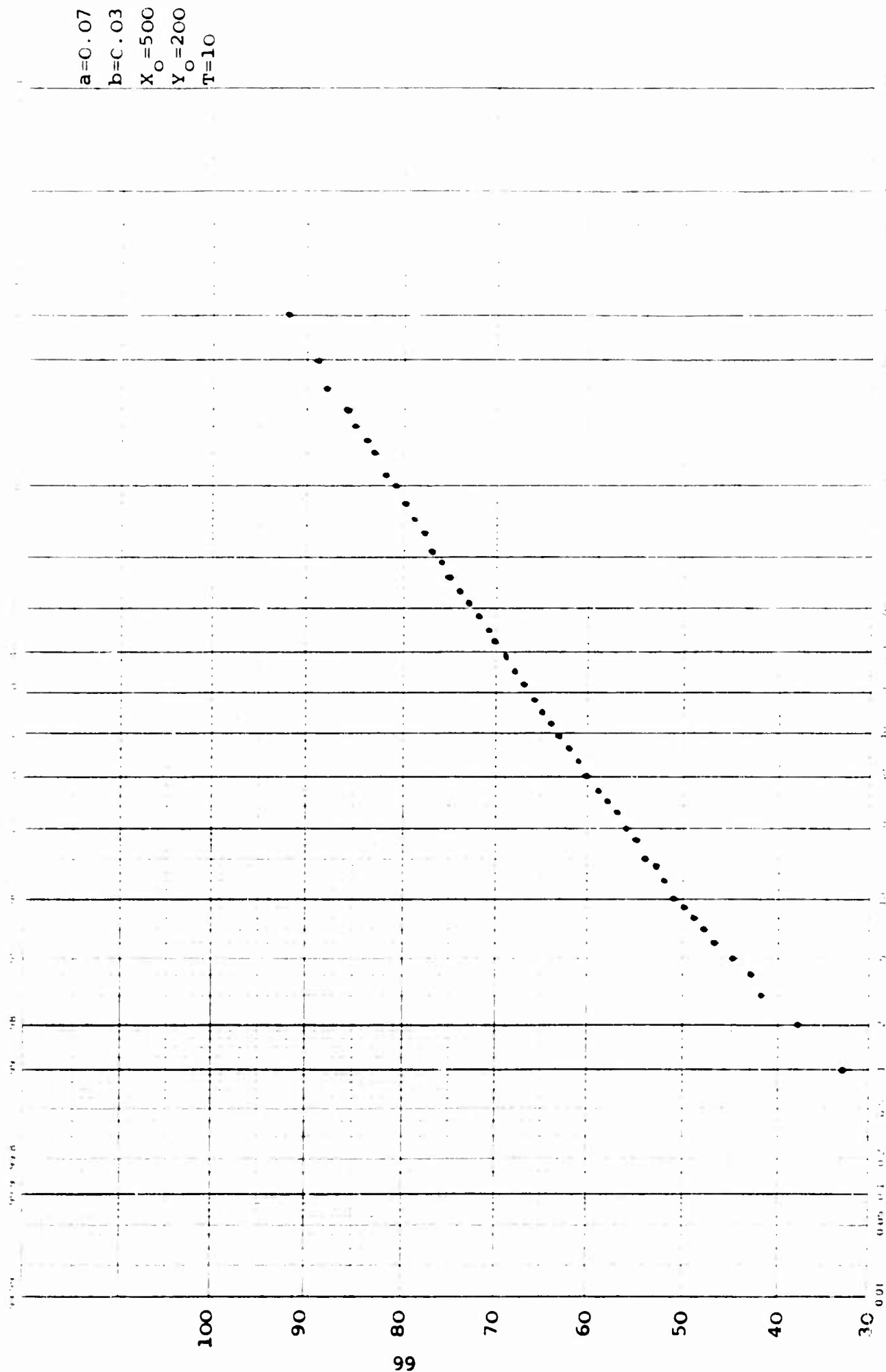


Figure 4.4.3a

X Marginal - Diffusion Square Law



Y Marginal - Diffusion Square Law

Figure 4.4.3b

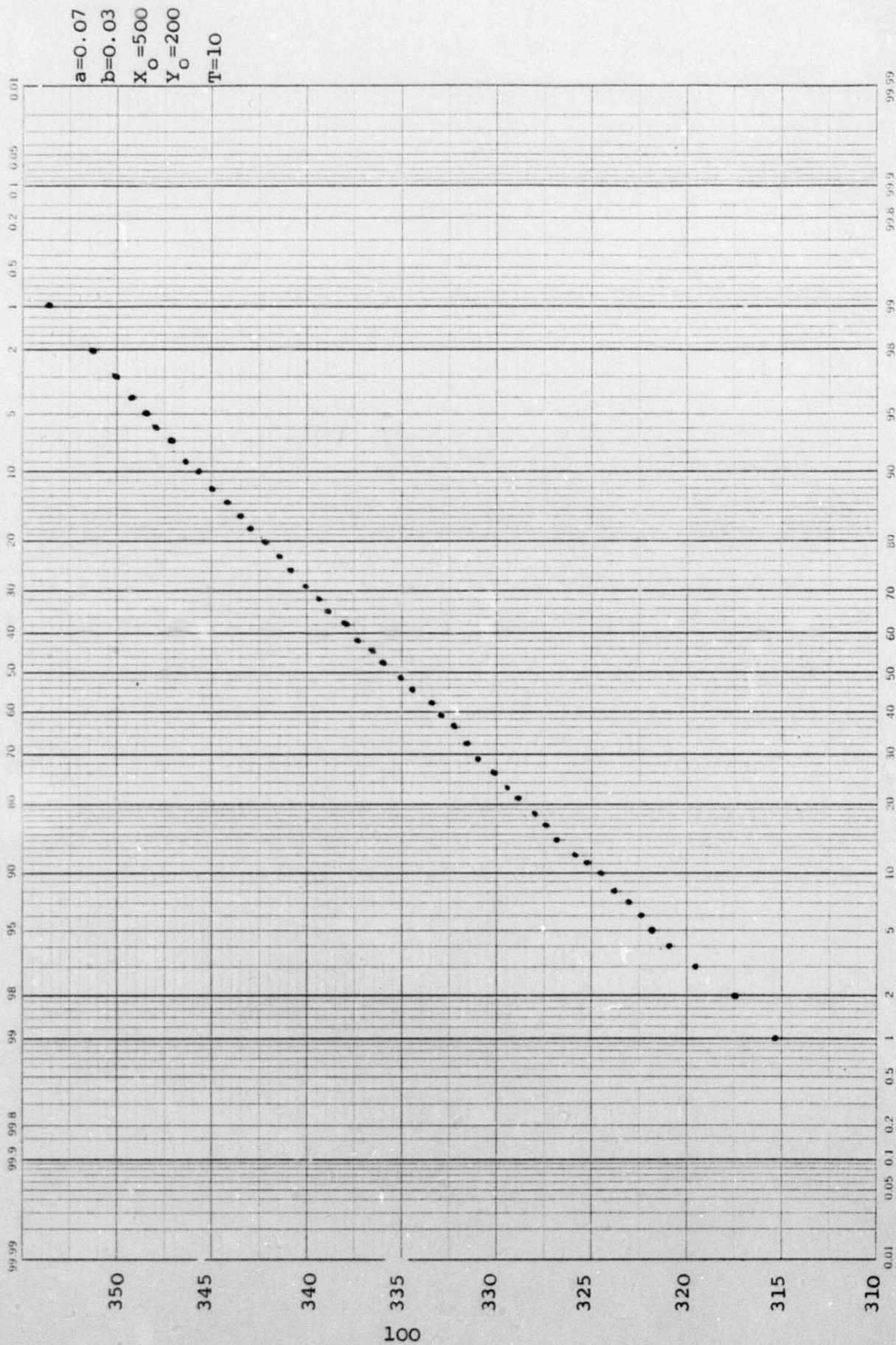


Figure 4.4.4a

Rotated X Marginal - Diffusion (45° Rotation) Square Law

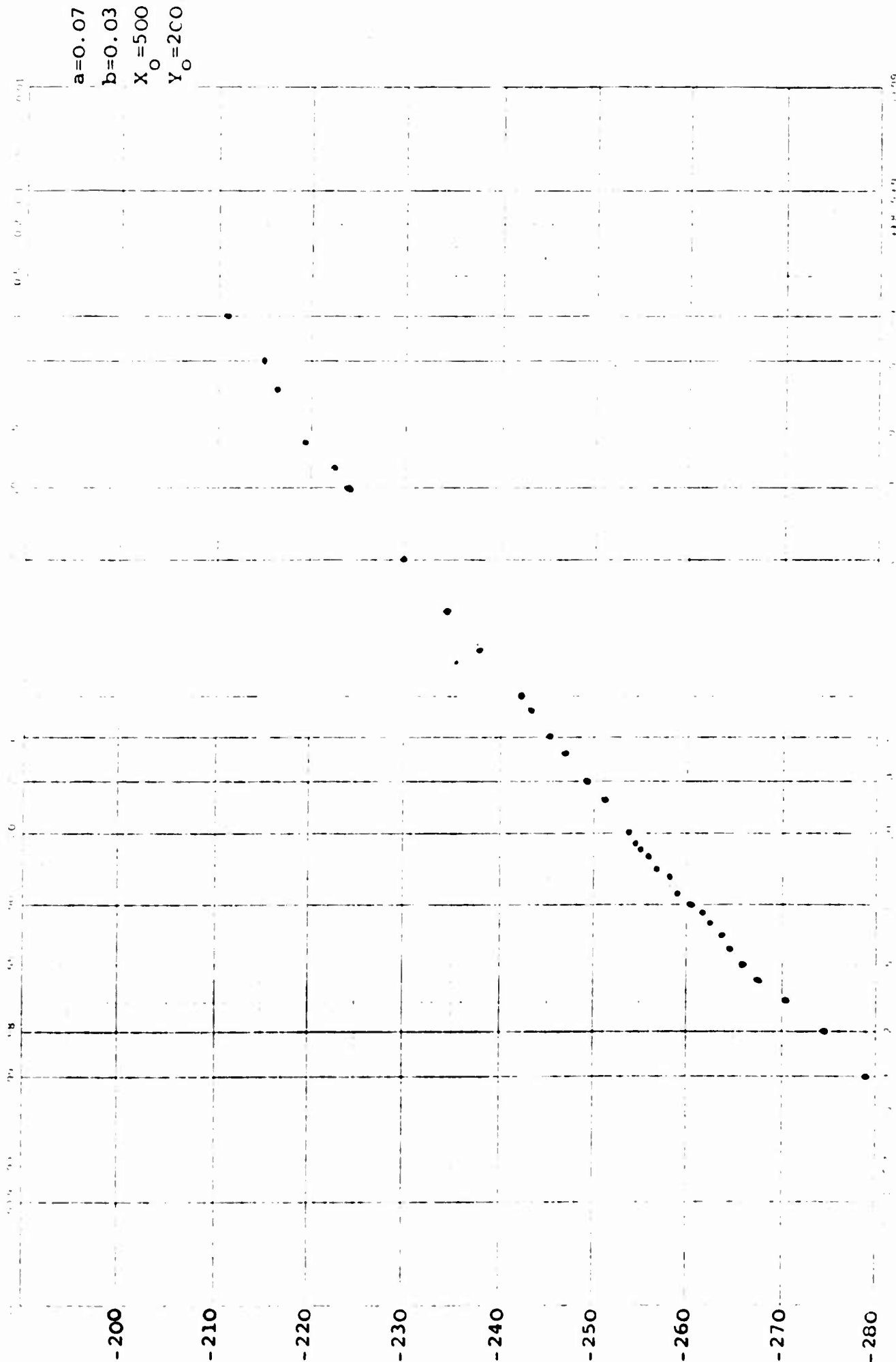


FIGURE 1. A. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 222. 223. 224. 225. 226. 227. 228. 229. 230. 231. 232. 233. 234. 235. 236. 237. 238. 239. 240. 241. 242. 243. 244. 245. 246. 247. 248. 249. 250. 251. 252. 253. 254. 255. 256. 257. 258. 259. 260. 261. 262. 263. 264. 265. 266. 267. 268. 269. 270. 271. 272. 273. 274. 275. 276. 277. 278. 279. 280. 281. 282. 283. 284. 285. 286. 287. 288. 289. 290. 291. 292. 293. 294. 295. 296. 297. 298. 299. 300. 301. 302. 303. 304. 305. 306. 307. 308. 309. 310. 311. 312. 313. 314. 315. 316. 317. 318. 319. 320. 321. 322. 323. 324. 325. 326. 327. 328. 329. 330. 331. 332. 333. 334. 335. 336. 337. 338. 339. 340. 341. 342. 343. 344. 345. 346. 347. 348. 349. 350. 351. 352. 353. 354. 355. 356. 357. 358. 359. 360. 361. 362. 363. 364. 365. 366. 367. 368. 369. 370. 371. 372. 373. 374. 375. 376. 377. 378. 379. 380. 381. 382. 383. 384. 385. 386. 387. 388. 389. 390. 391. 392. 393. 394. 395. 396. 397. 398. 399. 400. 401. 402. 403. 404. 405. 406. 407. 408. 409. 410. 411. 412. 413. 414. 415. 416. 417. 418. 419. 420. 421. 422. 423. 424. 425. 426. 427. 428. 429. 430. 431. 432. 433. 434. 435. 436. 437. 438. 439. 440. 441. 442. 443. 444. 445. 446. 447. 448. 449. 450. 451. 452. 453. 454. 455. 456. 457. 458. 459. 460. 461. 462. 463. 464. 465. 466. 467. 468. 469. 470. 471. 472. 473. 474. 475. 476. 477. 478. 479. 480. 481. 482. 483. 484. 485. 486. 487. 488. 489. 490. 491. 492. 493. 494. 495. 496. 497. 498. 499. 500. 501. 502. 503. 504. 505. 506. 507. 508. 509. 510. 511. 512. 513. 514. 515. 516. 517. 518. 519. 520. 521. 522. 523. 524. 525. 526. 527. 528. 529. 530. 531. 532. 533. 534. 535. 536. 537. 538. 539. 540. 541. 542. 543. 544. 545. 546. 547. 548. 549. 550. 551. 552. 553. 554. 555. 556. 557. 558. 559. 560. 561. 562. 563. 564. 565. 566. 567. 568. 569. 570. 571. 572. 573. 574. 575. 576. 577. 578. 579. 580. 581. 582. 583. 584. 585. 586. 587. 588. 589. 590. 591. 592. 593. 594. 595. 596. 597. 598. 599. 600. 601. 602. 603. 604. 605. 606. 607. 608. 609. 610. 611. 612. 613. 614. 615. 616. 617. 618. 619. 620. 621. 622. 623. 624. 625. 626. 627. 628. 629. 630. 631. 632. 633. 634. 635. 636. 637. 638. 639. 640. 641. 642. 643. 644. 645. 646. 647. 648. 649. 650. 651. 652. 653. 654. 655. 656. 657. 658. 659. 660. 661. 662. 663. 664. 665. 666. 667. 668. 669. 670. 671. 672. 673. 674. 675. 676. 677. 678. 679. 680. 681. 682. 683. 684. 685. 686. 687. 688. 689. 690. 691. 692. 693. 694. 695. 696. 697. 698. 699. 700. 701. 702. 703. 704. 705. 706. 707. 708. 709. 710. 711. 712. 713. 714. 715. 716. 717. 718. 719. 720. 721. 722. 723. 724. 725. 726. 727. 728. 729. 730. 731. 732. 733. 734. 735. 736. 737. 738. 739. 740. 741. 742. 743. 744. 745. 746. 747. 748. 749. 750. 751. 752. 753. 754. 755. 756. 757. 758. 759. 760. 761. 762. 763. 764. 765. 766. 767. 768. 769. 770. 771. 772. 773. 774. 775. 776. 777. 778. 779. 780. 781. 782. 783. 784. 785. 786. 787. 788. 789. 790. 791. 792. 793. 794. 795. 796. 797. 798. 799. 800. 801. 802. 803. 804. 805. 806. 807. 808. 809. 810. 811. 812. 813. 814. 815. 816. 817. 818. 819. 820. 821. 822. 823. 824. 825. 826. 827. 828. 829. 830. 831. 832. 833. 834. 835. 836. 837. 838. 839. 840. 841. 842. 843. 844. 845. 846. 847. 848. 849. 850. 851. 852. 853. 854. 855. 856. 857. 858. 859. 860. 861. 862. 863. 864. 865. 866. 867. 868. 869. 870. 871. 872. 873. 874. 875. 876. 877. 878. 879. 880. 881. 882. 883. 884. 885. 886. 887. 888. 889. 890. 891. 892. 893. 894. 895. 896. 897. 898. 899. 900. 901. 902. 903. 904. 905. 906. 907. 908. 909. 910. 911. 912. 913. 914. 915. 916. 917. 918. 919. 920. 921. 922. 923. 924. 925. 926. 927. 928. 929. 930. 931. 932. 933. 934. 935. 936. 937. 938. 939. 940. 941. 942. 943. 944. 945. 946. 947. 948. 949. 950. 951. 952. 953. 954. 955. 956. 957. 958. 959. 960. 961. 962. 963. 964. 965. 966. 967. 968. 969. 970. 971. 972. 973. 974. 975. 976. 977. 978. 979. 980. 981. 982. 983. 984. 985. 986. 987. 988. 989. 990. 991. 992. 993. 994. 995. 996. 997. 998. 999. 1000.

Comments: Once again, Tables 4.4.2a and 4.4.2b indicate the excellent agreement of the moments calculated from the diffusion model and the sample moments of the simulation. The normal plots also have the same general quality in the Square Law case as they do in that of the Linear Law, indicating a reasonably good normal fit in the center with some slightly more pronounced deviation in the tails. On the whole, the normal approximation seems to be rather a good one in the Square Law case as well.

The use of these normal approximations in the solution of a two stage decision problem is discussed in Section 4.5.

4.5. Solution of a Two Stage Decision Problem

The basic mathematical elements of a two stage combat decision problem are presented in Section 2.3. As stated there, the solution of such a problem requires knowledge of the force level distribution as an explicit function of time. The difficulties of deriving such distributions and the intractability of existing expressions has led to their approximation by the diffusion methods of Sections 4.2 and 4.3. These diffusion models, when combined with the martingale methods of Chapter 3, provide a means for solving the two stage problem.

Unfortunately, however, the nature of the expressions involved precludes an analytical solution and once again numerical techniques are required. This section will outline the elements of the numerical solution of the problem and present some selected results which were obtained using these techniques in the case of a Linear Law model.

The essence of the problem is to calculate the optimal initial force level, that is, the value of X_0 which makes the risk a minimum. The risk $\rho_2(X_0)$ is defined by equation (2.3.3) and is given by

$$\rho_2(X_0) = \int_{((X_T, Y_T))} \rho_2^T(X_0) dP_{X_0}^\theta(X_T, Y_T).$$

The function $\rho_2^T(X_0)$ is the risk incurred from making a force level decision X_0 at time 0, arriving at configuration (X_T, Y_T) at time T and proceeding optimally. This function is defined in turn by

$$\rho_2^T(X_0) = cX_0 + (X_0 - X_T) + \rho_1^*(X_T, Y_T)$$

where c is the cost of the initial forces and $\rho_1^*(X_T, Y_T)$ is the optimal risk attainable in a one stage problem with force configuration (X_T, Y_T) . (See Section 2.3.) The integration of $\rho_2^T(X_0)$ is performed with respect to the correct probability distribution of the force level configuration (X_T, Y_T) .

Since it is not possible to solve for the critical points of the $\rho_2(X_0)$ function in the usual way (by taking its derivative and solving for roots), it becomes necessary to search over the admissible values of X_0 in order to find the optimum. A basic requirement, then, is the ability to evaluate the risk function $\rho_2(X_0)$ for any possible value of X_0 .

The value of $\rho_2(X_0)$ depends on the distribution of the force level configuration at time T . The diffusion models allow approximation of such distributions by a bivariate normal distribution. The method of obtaining the correct moments for this normal approximation is discussed in some detail in Section 4.3.

Given the correct approximate bivariate normal distribution for (X_T, Y_T) , it remains only to calculate the integral of equation (2.3.3). This integral cannot be solved analytically and once again numerical techniques are employed. In this case, since the integral is a double integral, standard techniques of solution are somewhat limited. Perhaps the simplest and most practical method is that of Monte Carlo integration, a technique which, in this case, amounts to the estimation of the expected value of a random function by the arithmetic mean of a large random sample.

This technique was employed, using a pseudo-random normal generator developed by Kinderman and Ramage (1976), and a uniform generator developed by Lewis and Payne (1973).

A random sample of 500 (X_T, Y_T) points was taken for each initial X_0 value considered. A one stage problem, modified to take into account the presence of the X_T force in the manner discussed in Section 2.3, was then solved for each such point and thus $\rho_2^{T_i}(X_0)$ was calculated for i running from 1 to 500. Finally an estimate of $\rho_2(X_0)$ was obtained by using the sample mean

$$\rho_2(X_0) \approx \frac{1}{500} \sum_{i=1}^{500} \rho_2^{T_i}(X_0).$$

The accuracy of this estimate can be given in terms of its estimated standard deviation, Table 4.5.1 presents some sample values from these calculations.

Thus the optimal risk could be approximated for any value of X_0 . The optimal value of X_0 was then found by searching over those values of X_0 which were admissible solutions. In the Linear Law cases considered, the risk function for the two stage problem, as calculated by the above methods, proved to have relatively flat peaks and valleys. For this reason, there were sometimes several force levels with virtually the same risk.

Table 4.5.2 exhibits the results of the two stage procedure for some selected cases. Although only the actual optimal force level is listed, it should be kept in mind that in most cases these optimal values were not sharply defined.

A study of Table 4.5.2 and a comparison to the one stage results of Table 2.4.1 and Table 3.5.1 reveal some interesting points. First, as is to be expected, the risk of the two stage procedure is somewhat less than that of the one stage procedure. Also, the initial force levels tend to be less in the two-stage problem than the corresponding optimal levels for the one stage problem. Note that, in general, the actual values of the attrition constants a and b seem to have little effect on the optimal decision. An exception to this is the first group of cases in which the optimal level seems to decrease as the value of a and b increases. This same phenomenon occurs to a lesser degree in other cases as well. The lack of a sharp minimum makes such a result difficult to assess at this time. However, the question of whether some such effects are present is an intriguing one deserving of further investigation.

It can be seen from the above results that the diffusion approximation together with the martingale methods of Chapter 3 present a complete approach to the solution of the complex decision problems presented in Chapter 2. Extension of the methods of the two stage problem to solution of multi-stage ones is theoretically straightforward. Unfortunately, some rather serious problems in terms of actual computational complexity arise which require more efficient numerical techniques if larger problems are to be solved in a reasonable amount of computer time.

Table 4.5.1 - Sample Results of Two-Stage Optimization

Parameters: $Y_0 = 100$,
 $a = 0.0001$
 $b = 0.0001$
Cost of Initial Forces = 0.5
Cost of Reinforcements = 0.6,
Value of Victory = 500
 $T = 30$

Initial Force	Estimated Risk	Est. Standard Deviation of Estimate
110	-329.167	0.823
111	-330.679	0.834
112	-329.950	0.838
113	-330.871	0.824
114	-329.797	0.775
115	-330.106	0.824
116	-331.232	0.795
117	-330.897	0.771
118	-329.699	0.808
119	-331.339	0.758
120	-331.137	0.771
121	-331.187	0.776
122	-331.569	0.708
123	-330.771	0.715
124	-331.760	0.707
125	-331.373	0.706
126	-331.780	0.710
127	-330.822	0.694
128	-330.758	0.698
129	-331.210	0.644
130	-330.471	0.642

Table 4.5.2 - Some Numerical Results for the Two-Stage Decision

Problem

Cost of Initial Forces - 0.5

Cost of Reinforcements - 0.6

Time before Reinforcement - 30.0

Y_0	a	b	$p = \frac{b}{a+b}$	V	X	$\rho(X_0)$
100	0.0001	0.0001	0.5	500	120.0	-331.78
100	0.0005	0.0005	0.5	500	122.0	-336.18
100	0.0008	0.0008	0.5	500	108.0	-338.10
1000	0.00001	0.00001	0.5	5000	1105.0	-3434.72
1000	0.00005	0.00005	0.5	5000	1103.0	-3445.87
1000	0.00008	0.00008	0.5	5000	1104.0	-3447.43
100	0.00035	0.00015	0.3	500	282.0	-121.80
100	0.0007	0.0003	0.3	500	276.0	-125.27
1000	0.000035	0.000015	0.3	5000	2551.0	-1386.67
1000	0.00007	0.00003	0.3	5000	2552.0	-1393.19
100	0.00015	0.00035	0.7	500	55.0	-425.66
100	0.0003	0.0007	0.7	500	52.0	-427.10
1000	0.000015	0.000035	0.7	5000	488.0	-4321.47
1000	0.00003	0.00007	0.7	5000	476.0	-4325.66

Chapter 5

EXTENSIONS AND CONCLUSIONS

5.1. General Battle Termination Criteria

In the basic combat decision problem discussed in Section 2.1, the battle is assumed to continue until one side or the other is totally eliminated (that is, a "fight to the finish" criterion). Such an approach is, admittedly, a simplistic approximation to what is, in fact, a very complex process. The question of what alternate modeling techniques are better than the "fight to the finish" is a relatively open one. This section does not attempt to produce or endorse any specific criteria, rather it presents some intuitively reasonable characteristics of such criteria which would allow the martingale techniques of Chapter 3 to be employed with the same effectiveness as that found in the "fight to the finish" case.

In general, battle termination may be modelled by some bivariate function $B(\cdot, \cdot)$. Assume that $B(\cdot, \cdot)$ is continuous over the first quadrant of the Cartesian coordinate plane. The combat is considered to terminate when the trajectory of the (X, Y) (force level) combat process first crosses the curve described by $B(x, y) = 0$. (See Figure 5.1.1.) (Note that since (X, Y) is, strictly speaking, a discrete process, it is possible that the actual force levels upon termination of the combat process will not, in fact, be an element of the set describing the curve.) For simplicity, we shall assume, in

in addition, that $B(x,y) = 0$ defines y as a convex function of x .

Under the above conditions we can define three regions on the curve in a reasonable and intuitively satisfying manner. (See Figure 5.1.1.) The branch to the left of the point a , labeled Y WIN, represents the set of all force level configurations resulting in a termination of the battle with the side represented by the Y force level being victorious. Similarly, that branch of the curve to the right of the point b represents those force level configurations resulting in an X victory. The region between the two points a and b , labeled NO WIN, represents those force level configurations for which the combat terminates, but without a victor. (The possibility of such an occurrence will be considered no further here. It is included only for the sake of completeness.)

The general type of battle termination criteria described above allows full use of the martingale techniques of Chapter 3. Recall that the martingale methods are based on the following consideration. If the initial force levels X_0 and Y_0 are M and N respectively, then after a total of n transitions (n total casualties) the force levels X_n and Y_n must satisfy

$$X_n + Y_n + n = M + N.$$

Thus the martingale $K_n = K(X_n, Y_n)$ is a function of X_n alone (and implicitly of M , N , and n).

As demonstrated in Chapter 3, the distribution of K_{M+N} can be well approximated by a normal distribution. It was also shown that if either one of x or y is held fixed, the $K(x,y)$ function is monotonic in the other variable. (For example, if $K(x,y) = bx - ay$, for fixed y , K is an increasing function of x .) Due to this monotonicity property, it proved a relatively simple matter to use the normal approximation to the distribution of K_{M+N} to approximate probabilities of the form

$$P(X_f > x' | X \text{ wins})$$

where X_f represents the X force level upon the termination of the combat.

In order to implement the same type of procedure in the case of the more general battle termination criteria, we define the following sets:

$$T = \{(x,y) | B(x,y) = 0\}$$

$$T_X = \{(x,y) | B(x,y) = 0, X \text{ wins}\} \quad (5.1.1)$$

$$T_X(x') = \{(x,y) | B(x,y) = 0, X \text{ wins}, x \geq x'\}.$$

Note that $T_X(x') \subset T_X \subset T$. The set T represents what will be called the terminal points of the process. (As mentioned previously, it is possible that actual force levels at which the combat terminates, which are integer valued, may not satisfy $B(x,y) = 0$. However, the shorthand notation of (5.1.1) will be used for convenience.) The set T_X represents those terminal points for which X is victorious. Finally, $T_X(x')$ is the set

of terminal points for which a victorious X side has at least x' survivors.

Each of the sets of (5.1.1) induce a set of corresponding values of the function K . These sets may be defined as:

$$\begin{aligned} S &= \{K_{M+N}(x,y) \mid (x,y) \in T\} \\ S_X &= \{K_{M+N}(x,y) \mid (x,y) \in T_X\} \\ S_X(x) &= \{K_{M+N}(x,y) \mid (x,y) \in T_X(x')\}. \end{aligned} \quad (5.1.2)$$

Now if $K_{M+N}(x,y) \in S_X(x')$ (that is $(x,y) \in T_X(x')$) then we can calculate

$$P(T_X(x') \mid T_X) = P(X_f \geq x' \mid X \text{ wins}) = P(S_X(x') \mid S_X). \quad (5.1.3)$$

Therefore, in order for the martingale methods to be of use in the case of general battle termination criteria, the probability (5.1 3) must be readily calculable from the approximate normal distribution of K_{M+N} . In the "fight to the finish" criterion, this calculation is easy since the function $K(\cdot, \cdot)$ is monotonic as one proceeds in a counter-clockwise direction around the boundary (the Y and X axes). This monotonicity property may well be lacking in the case of a boundary function which, when considered as defining Y as a function of X , is not monotonic (Figure 5.1.1b). In such cases, calculation of $P(S_X(x') \mid S_X)$ may be difficult. The key points, however, are that calculation of that probability must be possible, and that $P(S_X(x') \mid S_X)$ must be the required quantity (that is

$K_{M+N}(x,y) \in S_X(x')$ must be equivalent to $(x,y) \in T_X(x')$.

The most immediate weakening of the "fight to the finish" requirement satisfies the above conditions. Suppose the combat terminates when the trajectory crosses either one of two straight lines parallel to the coordinate axes ($X = \alpha X_0$, $Y = \beta Y_0$ for some $0 \leq \alpha, \beta < 1$. See Figure 5.1.2).^{*} In this case, the methods employed are almost identical to those of Chapter 3. Indeed, in the Linear Law case, the problem can be transformed immediately into a "fight to the finish" type problem by simply adjusting the initial starting configuration from (X_0, Y_0) to $(X_0(1 - \alpha), Y_0(1 - \beta))$. The case of the Square Law requires a more delicate analysis, but the methods used in this case can also be easily modified to take the new set of terminal points into account.

The efficacy of the martingale methods of Chapter 3 in the analysis of the combat models with various battle termination dynamics is thus seen to be somewhat dependent on the specific model of termination employed. Terminal curves similar to the coordinate axes can be employed with a minimal modification to the methods of Chapters 3 and 4. More exotic curves, however, may render the martingale technique more difficult, if not impossible, to employ. These problems of analysis should, therefore, be considered when attempting to model battle termination.

^{*} This is the type of battle termination criteria employed in U.S. Army Field Manual 105-5, Maneuver Control.

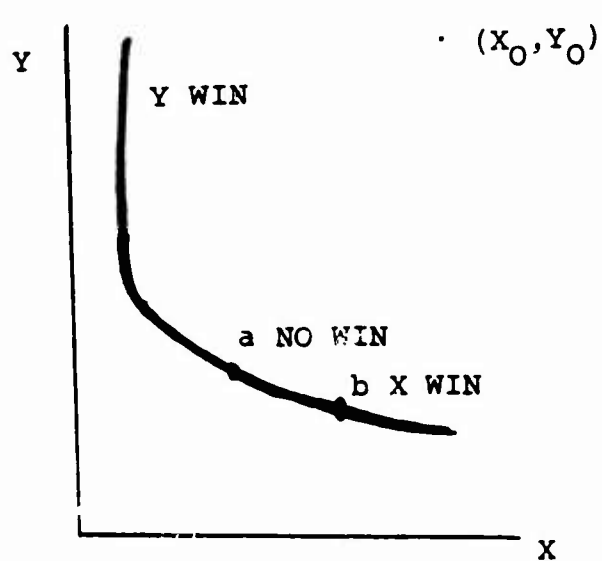


Figure 5.1.1a

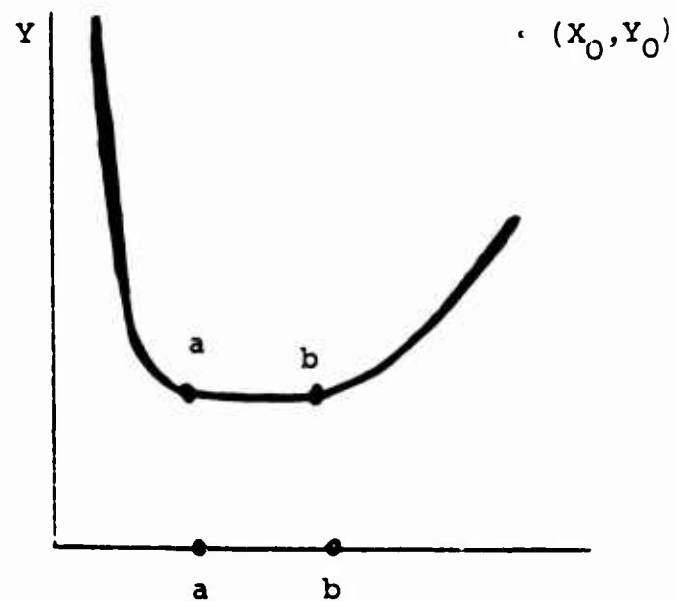


Figure 5.1.1b

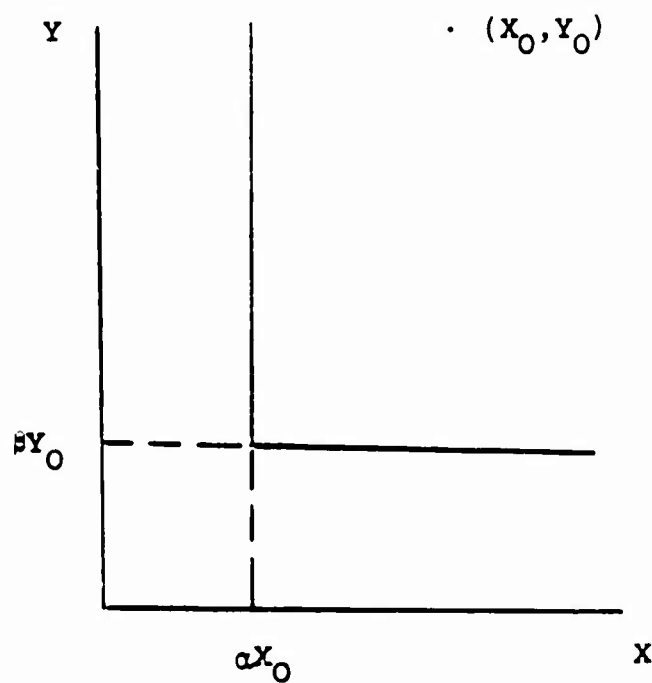


Figure 5.1.2

5.2. Non-Trivial Prior Distributions and Uncertainty in Combat

One of the criticisms leveled at Lanchester's original models is that they assume full information to be available about the numerical strength and effectiveness of the enemy force as well as the friendly force. (See Section 1.2.) Although the decision theoretic framework introduced in Chapter 2 made provision for uncertainty about these, as well as other variables, the models and results presented in Chapters 3 and 4 have made use of trivial, or point, probability distributions which merely represent full information (or at least the belief that one has full information!). An investigation of non-trivial distributions is, therefore in order.

Military commanders in combat situations are plagued with myriad uncertainties. The most pronounced are generally concerned with the capabilities and intentions of the enemy, the strength of the enemy force, its morale, its dispositions. He will also be unsure, though usually to a lesser extent, of his own forces, their morale, their determination, and his ability to control them and accomplish his objectives once the fighting begins.

A conscientious commander will do all in his power to obtain the most recent and most reliable information about all aspects of a forthcoming engagement. He will consult his subordinate officers about the condition of his own units and inspect the troops as much as possible. He will attempt to pierce the so-called "fog of war" surrounding the enemy, with any means at his disposal. Aerial reconnaissance, radio monitoring, foot patrols

and other methods will be employed to gather as much information as possible about the strength, location and intention of the enemy force.

Yet, even the most elaborate intelligence gathering efforts will often provide only very limited information and are certainly not available without some cost: men, materiel and time may be lost, the element of surprise, often so important in combat, may be given up if the level of intelligence activity becomes too high. In circumstances when time is limited or other conditions prohibit the extensive gathering and processing of information, the commander must act according to his best estimate of the true state of affairs, based on what limited hard facts are available and on his own past experience and beliefs.

In such situations, the beliefs and opinions of the commander may be thought of as forming a basis for his subjective probability distribution on the initial state of the combat situation he is facing. The numerical strength of the enemy is seldom known with any great precision; indeed, in some cases even friendly strength is uncertain. Even more variable and difficult to estimate is the combat effectiveness of the enemy forces; their morale, supply and command situations, as well as many other similar factors, are often almost impossible to assess with accuracy from a purely objective point of view. In fact, the art of the good commander lies in his ability to assess just such factors accurately and to employ his own limited resources in such a manner that the enemy weaknesses are exploited and his own strengths accentuated.

As a first step in introducing some of these uncertain factors into combat models in a more explicit manner, the next section of this chapter will deal with a specific simple case. All quantities are assumed to be fixed and known exactly except for the numerical strength, Y_0 , of the enemy force. The commander is assumed to have formulated a probability distribution, $G(\cdot)$, over the possible strengths of the enemy. This distribution represents his knowledge and opinions about the enemy force level. The discussion of this problem is presented through the use of an example, based on the Lanchester Linear Law model, so familiar by now, outlined in Section 3.3.

5.3. Examples of the Use of Subjective Distributions

If X_0 is the initial friendly force level employed in a one stage decision problem as outlined in Section 2.1, then the risk of X_0 , assuming all other variables to be fixed and known, is given by

$$\rho(X_0) = cX_0 + (X_0 - E(X_f|X_0)) - VP(X \text{ wins}|X_0).$$

The risk function is implicitly a function of Y_0 , and if Y_0 is not known we write, for a given value of Y_0 ,

$$\rho(X_0|Y_0) = cX_0 + (X_0 - E(X_f|X_0, Y_0)) - VP(X \text{ wins}|X_0, Y_0).$$

If $G(\cdot)$ is a probability distribution over the possible values of Y_0 , then

$$\begin{aligned} \rho(X_0) &= \int_{\{Y_0\}} \rho(X_0|Y_0) dG(Y_0) = cX_0 + X_0 - \int_{\{Y_0\}} E(X_f|(X_0, Y_0)) dG(Y_0) \\ &\quad - V \int_{\{Y_0\}} P(X \text{ wins}|(X_0, Y_0)) dG(Y_0). \end{aligned}$$

The calculation of $\rho(X_0)$ thus requires the evaluation of two integrals

$$\int_{\{Y_0\}} E(X_f|(X_0, Y_0)) dG(Y_0) \tag{5.3.1}$$

and

$$\int_{\{Y_0\}} P(X \text{ wins}|(X_0, Y_0)) dG(Y_0). \tag{5.3.2}$$

Since

$$\begin{aligned} E(X_f | (X_0, Y_0)) &= E(X_f | (X_0, Y_0), X \text{ wins}) P(X \text{ wins} | (X_0, Y_0)) \\ &\quad + E(X_f | (X_0, Y_0), X \text{ lose}) P(X \text{ lose} | (X_0, Y_0)), \end{aligned}$$

and

$$E(X_f | (X_0, Y_0), X \text{ lose}) = 0 \text{ (fight to the finish),}$$

we find

$$E(X_f | (X_0, Y_0)) = E(X_f | (X_0, Y_0), X \text{ win}) P(X \text{ win} | (X_0, Y_0)). \quad (5.3.3)$$

Thus, upon substituting (5.3.3), expression (5.3.1) becomes

$$\int_{\{Y_0\}} E(X_f | (X_0, Y_0), X \text{ wins}) P(X \text{ wins} | X_0, Y_0) dG(Y_0). \quad (5.3.4)$$

In order to calculate $\rho(X_0)$, we must now evaluate expressions (5.3.2) and (5.3.4).

In the case of a Linear Law model, the martingale is formed from the function $K(x, y) = px - qy$ as discussed in Section 3.5. It was shown in that section that

$$E(X_f | X_0, Y_0, X \text{ win}) = \frac{\mu}{p} + \left[\frac{q}{p} \Phi(\mu/\sigma) \right] / \Phi(\mu/\sigma)$$

and

$$P(X \text{ wins} | (X_0, Y_0)) = \Phi(\mu/\sigma)$$

where

$$\mu = pX_0 - qY_0, \quad \sigma^2 = \begin{cases} pX_0 & \text{if } \mu < 0 \\ qX_0 & \text{if } \mu > 0. \end{cases}$$

Thus expression (5.3.4) becomes

$$\begin{aligned} & \int_{\{Y_0\}} E(X_f | (X_0, Y_0), X \text{ wins}) P(X \text{ wins} | (X_0, Y_0)) dG(Y_0) \\ &= \int_{\{Y_0\}} \left[\frac{\mu}{p} + \frac{\sigma}{p} \frac{\phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \right] \Phi(\mu/\sigma) dG(Y_0) = \frac{1}{p} \int_{\{Y_0\}} [\mu \Phi(\mu/\sigma) + \sigma \phi(\mu/\sigma)] dG(Y_0) \\ &= \frac{1}{p} \int_{pX_0/q}^{\infty} \left[(pX_0 - qY_0) \Phi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) + \sqrt{pX_0} \phi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) \right] dG(Y_0) \\ &+ \frac{1}{p} \int_{-\infty}^{pX_0/q} \left[(pX_0 - qY_0) \Phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) + \sqrt{qY_0} \phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) \right] dG(Y_0). \end{aligned} \quad (5.3.5)$$

Similarly (5.3.2) is given by

$$\int_{\{Y_0\}} \Phi(\mu/\sigma) dG(Y_0) = \int_{-\infty}^{pX_0/q} \Phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) dG(Y_0) + \int_{pX_0/q}^{\infty} \Phi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) dG(Y_0) \quad (5.3.6)$$

Combining expressions (5.3.5) and (5.3.6) the risk of X_0 is given by

$$\begin{aligned}
\rho(X_0) = & cX_0 + X_0 - \frac{1}{p} \int_{-\infty}^{pX_0/q} \left[(pX_0 - qY_0) \Phi \left(\frac{pX_0 - qY_0}{\sqrt{qY_0}} \right) \right. \\
& \left. + \sqrt{qY_0} \varphi \left(\frac{pX_0 - qY_0}{\sqrt{qY_0}} \right) \right] dG(Y_0) \\
& - \frac{1}{p} \int_{pX_0/q}^{\infty} \left[(pX_0 - qY_0) \Phi \left(\frac{pX_0 - qY_0}{\sqrt{pX_0}} \right) + \sqrt{pX_0} \varphi \left(\frac{pX_0 - qY_0}{\sqrt{pX_0}} \right) \right] dG(Y_0) \\
& - v \left[\int_{-\infty}^{pX_0/q} \Phi \left(\frac{pX_0 - qY_0}{\sqrt{qY_0}} \right) dG(Y_0) + \int_{pX_0/q}^{\infty} \Phi \left(\frac{pX_0 - qY_0}{\sqrt{pX_0}} \right) dG(Y_0) \right].
\end{aligned}
\tag{5.3.7}$$

We wish to find an X_0 such that the value of (5.3.7) is a minimum. Although the function to be minimized is well defined, one may encounter substantial numerical problems in finding its minimum.

One possible approach to the solution of the problem makes use of a continuous approximation to a discrete distribution. (A similar philosophy is seen in the use of the diffusion approximation of Chapter 4. It was shown in this case that a continuous approximation to the discrete state space works well.) Specifically, assume that $G(Y_0)$ has a density function with respect to Lebesgue measure so that $dG(Y_0) = g(Y_0)dy_0$, where $g(Y_0)$ is the approximating density function. This approach seeks to find the optimal X_0 through the usual method of differentiating (5.3.7) as a function of X_0 and solving for roots. Unfortunately, the complexity of

(5.3.7) makes this quite a difficult analytic problem. This derivative can, in fact, be computed explicitly to be

$$\begin{aligned}
 \rho'(X_0) = & c + 1 - \int_{-\infty}^{pX_0/q} \Phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) g(Y_0) dY_0 \\
 & - \int_{pX_0/q}^{\infty} \Phi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) g(Y_0) dY_0 - \frac{p}{2} \int_{pX_0/q}^{\infty} \sqrt{\frac{p}{X_0}} \varphi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) g(Y_0) dY_0 \\
 & - v \left\{ \int_{-\infty}^{pX_0/q} \varphi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right) \frac{p}{\sqrt{qY_0}} g(Y_0) dY_0 \right. \\
 & \quad + \int_{pX_0/q}^{\infty} \left[\frac{p}{2} (pX_0)^{-1/2} + \frac{p}{2} qY_0 (pX_0)^{-3/2} \right] \\
 & \quad \left. \varphi\left(\frac{pX_0 - qY_0}{\sqrt{pX_0}}\right) g(Y_0) dY_0 \right\} .
 \end{aligned}
 \tag{5.3.8}$$

Clearly, solving for the roots of (5.3.8) as a function of X_0 , will, in general, present some major practical difficulties.

An alternative to the technique discussed above would make use of equation (5.3.7) directly by searching over admissible (and reasonable) values of X_0 to find an optimum. As long as $\rho(X_0)$ is reasonably well behaved, this technique may be successful. If, however, an exhaustive or nearly exhaustive search must be

made over many values of X_0 , the cost of this procedure in terms of computing time may easily become very high.

Other possibilities exist. If the distribution G places all of its weight on a very small number of points, then the integrals of expression (5.3.7) are actually sums of only a few terms. Further approximations (such as the Mill's ratio approximation to the normal distribution function) may be employed to simplify the expressions. In general, however, the solution of the decision problem based on the risk function (5.3.7) remains an important topic requiring further serious research.

5.4. Topics for Further Research

Bayesian analysis of decision problems arising in combat is a topic which, up to now, has been barely explored. This thesis presents only the first steps towards such an analysis: the formulation of the general problem, and its partial solution for a simple model. Much remains to be investigated.

Section 5.3 presented an example, in general terms, of a subjective probability distribution over the enemy force level. The question of what type of distribution is reasonable for such a variable, and the qualities it should have, remain to be explored. The question of the formulation of distributions for the other variables, such as enemy effectiveness, terrain and weather effects, etc., and their interactions remains an even more open and difficult topic for analysis.

The use of intelligence about the enemy position and strength as well as other information obtained during the course of a battle should clearly play a role in any subsequent decisions a commander may make. The mechanics of processing such information to update the commander's prior distribution to a posterior distribution must be considered, and is, in fact, essential to a Bayesian analysis of the multi-stage problem.

Perhaps of more direct and immediate significance than the topics discussed above is the question of the sensitivity of the decisions based on the analytic methods advocated here. This question is especially important for the analysis of the decision problem based on point priors. Indeed, if slight deviations from the specified parameters results in widely varying decisions and

risks, the importance of finding a solution for the non-trivial problem is heightened. Similarly, in the case of such non-trivial prior distributions, sensitivity of results to slight deviations in the prior can make care in its accurate definition of great importance. Fortunately, the results of the numerical studies done so far show rather flat peaks in the risk function. In such a case, solution for the true optimum level is not so urgent, as any point in the neighborhood of the optimum will give virtually the same risk. This fact may alleviate some of the sensitivity problems. However, much remains to be done in this area.

Many interesting and important topics, some of which have been only slightly touched on above, remain to be explored. The extension of the diffusion and martingale methods, used to solve the two stage problem, to the development of optimal or near-optimal decisions for multi-stage problems is an open and challenging field for further investigation, and may be amenable to the application of some sort of control theory. (See Taylor (1973, 1974) for examples of the use of control theory with deterministic models.) An interesting question along these lines is under what conditions a myopic rule may be optimal or nearly optimal in a multi-stage problem.

Also of interest is the formulation and solution of a decision problem of the optimal resource allocation type. Suppose that a commander has some fixed force, say N , available, and he must allocate forces to each of k combat situations. He will seek

to find an allocation vector (n_1, n_2, \dots, n_k) subject to the constraints $n_i \geq 0, \forall i, \sum_{i=1}^k n_i = N$. If $\rho(n_1, n_2, \dots, n_k)$ is the total risk of an allocation of force n_i to combat i , $i = 1, 2, \dots, k$ then he wishes to choose a vector which minimizes this risk. If each of the combats is assumed to be in the form of the standard one-stage decision problem, then the methods developed above may prove useful to the solution of such a problem, and further work along these lines is indicated.

Another important and fascinating problem arises when the inherent two-sided nature of a combat action is considered. A battle might be formulated in terms of a two-person game. As noted in Section 1.4, some work has been done along these lines, but not employing the decision theoretic models introduced above. If the two opposing "players", or commanders, are assumed to make their decisions in accordance with the decision model proposed here, then an analysis of the effects and interactions of their various cost and reward structures, as well as the comparative effects of their information bases (prior distributions), should prove most interesting.

The list of further topics for research is limited only by one's imagination. This thesis has attempted to introduce a methodology and to use the methodology to solve some basic problems based on the simpler models already in existence. The same problems may be analyzed based on more complex models. Those which immediately come to mind are the cases of non-homogeneous forces and asymmetric attrition structures. However, other

effects, such as the impact of improved intelligence (sharper subjective distributions), command and control problems, and uncertain weapon effectiveness are important and deserving of analysis. This list is, of course, far from complete, but it can serve as a reasonable starting point for further research and analysis.

5.5. Summary

This thesis has attempted to extend the analysis of stochastic conflict models of the Lanchester type beyond the stage of mere modeling. To this end, the framework of statistical decision theory was adapted to a simplified combat situation: the military decision maker, or commander, was now faced with making decisions within the context of a suitable cost and reward structure. The decision problems which resulted from this structure were basically in the form of standard one-stage and multi-stage decision problems.

The one-stage decision problem, which required only that the commander choose the amount of force to employ in the combat prior to its inception, proved amenable to a complete solution through the use of martingale central limit theorems in a manner based on the work of Watson (1976). These methods provide a conceptually straightforward technique of calculating, approximately, all quantities necessary to solve the one-stage decision problem by use of the normal distribution.

Although continuous (real) time played little direct role in the one-stage decision problem, the nature of the multi-stage problem dictated that time would be of major importance in its analysis. The essence of the problem lies in the fact that reinforcements may become available to the commander over time. Because of this fact, it is necessary for the commander to know the force level distribution as an explicit function of time. The use of a diffusion approximation to the combat process allowed the approximation of this distribution, again, an approximation based on the normal distribution. This diffusion model, coupled

with the martingale methods, allowed solutions to be obtained for a two stage decision problem. Problems with more than two stages are easy to conceptualize, however their solution is still some way off. The diffusion approximation seems to be a most promising tool for use in solving such problems.

To provide some empirical support of the accuracy and utility of the martingale and diffusion methods of analysis, several numerical studies were undertaken. The results of these studies proved to be very encouraging. On the whole, the normal approximations were seen to be rather good ones except, perhaps, in the extreme tails. This fact, as well as the reasonable forms of the optimal decisions based on these approximations, gives some rather strong support to the efficacy of the proposed methodologies in the solution of the decision problems. In addition, the diffusion approximation provides an important contribution to the study of attrition processes of the Lanchester type in the continuous time setting.

In conclusion, therefore, this thesis serves as a foundation, a starting point for the further analysis of Lanchester type attrition models, and, more importantly the use of such models in the formulation and solution of the many decision problems which arise in combat situations.

APPENDIX: Proof that the Watson Square Law Martingale Satisfies
the Conditions of Scott's Martingale Central Limit
Theorem

To prove that Watson's Square Law martingale satisfies Scott's Central Limit Theorem we consider a triangular array in which the elements of the n th row are given by

$$S_0(n)S_1(n)\dots S_{N_n}(n) \quad \text{where } N_n \rightarrow \infty \text{ as } n \rightarrow \infty,$$

and in which $\{S_k(n), \mathcal{F}_k(n); 0 \leq k \leq N_n\}$ is a martingale sequence (see Section 3.2). Suppose that the initial force levels corresponding to the n th row are given by $(N\delta, N\epsilon)$ where δ and ϵ are known constants and $N\delta$ and $N\epsilon$ are integers. Let $N_n = N\delta + N\epsilon$ and $\mu_n = K(X_0, Y_0) = bX_0 - aY_0$ for attrition parameters a and b and $X_0 = N\delta$, $Y_0 = N\epsilon$. (Square Law.)

The row elements are defined by $S_k(n) = K(X_k, Y_k) - \mu_n$, $0 \leq k \leq N_n$, that is, the value of $K - \mu_n$ after k transitions have been made in the actual combat process. Then $\langle S_k(n) \rangle$ is a mean zero martingale sequence. Define $X_k(n) = S_k(n) - S_{k-1}(n)$ to be the increments of the process. ($X_k(n)$ should not be confused with X_k , the X force level after k transitions.) In this case, the possible values of $X_k(n)$ depend on the force levels prior to the k th transition. If these force levels are x and y , then $X_k(n)$ takes on the values $-bx$ and $+ay$ with probabilities $\frac{ay}{ay + bx}$ and $\frac{bx}{ay + bx}$ respectively.

We wish to show that, properly scaled, $S_{N_n}(n)$ will converge

weakly to a normal distribution as $n \rightarrow \infty$ by invoking Scott's triangular array martingale central limit theorem. (See Section 3.2.) Scott's theorem is satisfied if:

$$s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{P} t \text{ as } n \rightarrow \infty \quad (\text{A.1})$$

and

$$s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n) \xrightarrow{P} 0 \text{ as } n \rightarrow \infty \quad (\text{A.2})$$

where $s_k^2(n) = \text{Var } S_k(n)$, $0 \leq k \leq N_n$, and

$$m_n(t) = \max\{m \leq N_n \mid s_m^2(n) \leq t s_{N_n}^2(n)\} \text{ for } t \in [0,1].$$

The proof that the Square Law martingale satisfies (A.1) and (A.2) follows the same lines as the proof for the Linear Law martingale as shown in Section 3.3. We make use of the Watson (1976) approximation to the variance $s_{N_n}^2(n)$ which indicates that $s_{N_n}^2(n) = O(N^3)$.

The proof of (A.2) is immediate. Since the largest value of $X_k(n)$ must occur for the first transition in either the X or the Y direction,

$$s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n) = \begin{cases} s_{N_n}^{-2}(n) N^2 \delta^2 \\ \text{or} \\ s_{N_n}^{-2}(n) N^2 \epsilon^2. \end{cases}$$

Since Watson's approximation indicates that $s_{N_n}^2(n) = O(N^3)$, both $s_{N_n}^{-2}(n)N_n^{2/2}$ and $s_{N_n}^{-2}(n)N_n^2\epsilon^2$ converge to zero as $n \rightarrow \infty$, and so (A.2) is satisfied.

The proof of (A.1) follows the same lines as the proof of expression (3.3.2), and is based on Scott's Lemma 10. This proof requires some preliminary results.

Lemma 1.

$$s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n) \xrightarrow{P} 1. \quad (\text{A.3})$$

Proof: We wish to show

$$P(|s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n) - 1| > 0) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consider

$$s_{N_n}^2(n) = \text{Var } S_{N_n}(n) = E[(\sum_{k=1}^{N_n} X_k(n))^2]$$

since $E(S_{N_n}(n)) = 0$.

$$\begin{aligned} E[(\sum_{k=1}^{N_n} X_k(n))^2] &= E[\sum_{k=1}^{N_n} X_k^2(n) + 2 \sum_{i < j}^{N_n} X_i(n) X_j(n)] \\ &= E(\sum_{k=1}^{N_n} X_k^2(n)) + 2E[\sum_{i < j}^{N_n} X_i(n) X_j(n)]. \end{aligned}$$

Now,

$$E\left[\sum_{i < j} X_i(n) X_j(n)\right] = E\left\{E\left[\sum_{i < j} X_i(n) X_j(n) \mid \mathfrak{F}_i(n)\right]\right\}.$$

But

$$\begin{aligned} E[X_i(n) X_j(n) \mid \mathfrak{F}_i(n)] &= X_i(n) E(X_j(n) \mid \mathfrak{F}_i(n)) \\ &= X_i(n) [E(S_j(n) \mid \mathfrak{F}_i(n)) - E(S_{j-1}(n) \mid \mathfrak{F}_i(n))] \\ &= X_i(n) [S_i(n) - S_i(n)] = 0. \end{aligned}$$

Thus $s_{N_n}^2(n) = E\left(\sum_{k=1}^{N_n} X_k^2(n)\right)$ and so

$$E[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] = 1.$$

By the Chebyshev inequality

$$P(|s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n) - 1| > \epsilon) \leq \text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] / \epsilon^2.$$

Therefore in order to prove (A.3) it is sufficient to show

$$\text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(Recall from the definition of the triangular array that $n \rightarrow \infty$ if and only if $N \rightarrow \infty$.)

Lemma 2.

$$\text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{A.4})$$

Proof: Notice that

$$\text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} X_k^2(n)] = s_{N_n}^{-4}(n) \text{Var}(\sum_{k=1}^{N_n} X_k^2(n)).$$

Since $s_{N_n}^2(n) = O(N^3)$, it follows that $s_{N_n}^4(n) = O(N^6)$ and in order to demonstrate (A.4), it will be sufficient to show that

$$\text{Var}(\sum_{k=1}^{N_n} X_k^2(n)) = O(N^5).$$

To prove the latter, notice that

$$\text{Var}(\sum_{k=1}^{N_n} X_k^2(n)) = \sum_{k=1}^{N_n} \text{Var}(X_k^2(n)) + \sum_{i=1}^{N_n} \sum_{j \neq i}^{N_n} \text{Cov}(X_i^2(n), X_j^2(n)).$$

Consider first $\sum_{k=1}^{N_n} \text{Var}(X_k^2(n))$. For any $0 \leq k \leq N_n$,

$\text{Var} X_k^2(n) \leq E[X_k^4(n)]$. The possible values of $X_k(n)$ are of order at most N , so that

$$\text{Var} X_k^2(n) \leq O(N^4),$$

and

$$\sum_{k=1}^{N_n} \text{Var} X_k^2(n) \leq O(N^5),$$

since $N_n = O(N)$. It remains only to show:

Lemma 3.

$$\sum_i \sum_j \text{Cov}(X_i^2(n), X_j^2(n)) \leq O(N^5).$$

Proof: Consider first the $\text{Cov}(X_i^2, X_{i+1}^2)$ for any i . (The argument n of $X_i(n)$ is deleted in the following. Once again these X_i should not be confused with force levels.) To simplify notation only, let us refer to this as $\text{Cov}(X_1^2, X_2^2)$. Suppose that prior to the increment X_1 (that is, prior to the i th transition) the force level configuration is (x, y) where both x and y are of order $O(N)$. We wish to show

$$|\text{Cov}(X_1^2, X_2^2)| \leq O(N^3).$$

Refer to Figure A.1 for the possible configurations of X_1 and X_2 .

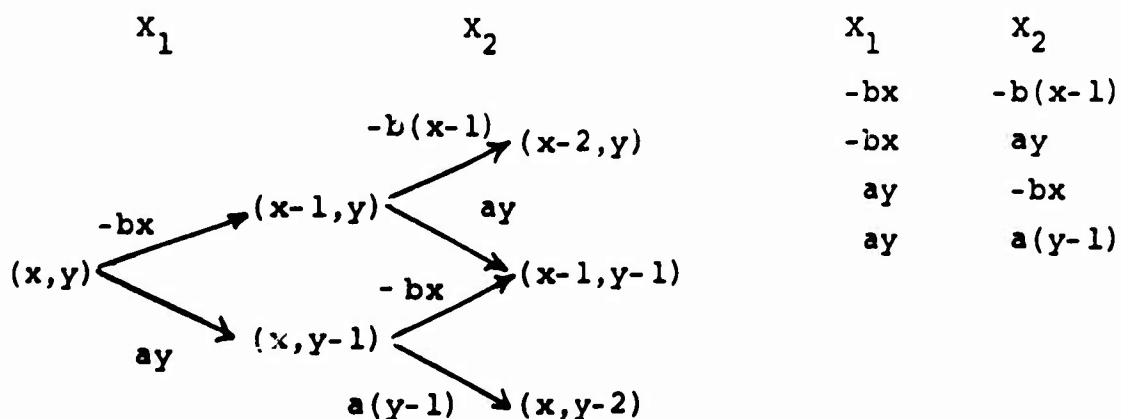


Figure A.1

A simple calculation gives

$$E(X_1^2, X_2^2) = \frac{b^4 x^2 (x-1)^2 a^2 y^2}{(ay + bx)(ay + b(x-1))} + \frac{a^4 y^2 (y-1)^2 b^2 x^2}{(ay + bx)(a(y-1) + bx)} \\ + a^2 b^2 x^2 y^2 \left[\frac{ayb(x-1)}{(ay + bx)(ay + b(x-1))} + \frac{bxa(y-1)}{(ay + bx)(a(y-1) + bx)} \right],$$

$$E(X_1^2) = abxy,$$

and

$$E(X_2^2) = E[X_2^2 | X_1 = -bx]P(X_1 = -bx) + E[X_2^2 | X_1 = ay]P(X_1 = ay) \\ = \frac{ay}{ay + bx} \left[\frac{b^2(x-1)^2 ay}{ay + b(x-1)} + \frac{a^2 y^2 b(x-1)}{ay + b(x-1)} \right] \\ + \frac{bx}{ay + bx} \left[\frac{b^2 x^2 a(y-1)}{a(y-1) + bx} + \frac{a^2 (y-1)^2 bx}{a(y-1) + bx} \right].$$

So,

$$\text{Cov}(X_1^2, X_2^2) = \frac{abxy}{ay + bx} \left[\underbrace{\frac{b^3 x(x-1)^2 ay}{ay + b(x-1)}}_{(1)} + \underbrace{\frac{a^3 y(y-1)^2 bx}{a(y-1) + bx}}_{(2)} + \underbrace{\frac{a^2 b^2 xy^2 (x-1)}{ay + b(x-1)}}_{(3)} \right. \\ \left. + \underbrace{\frac{a^2 b^2 x^2 y(y-1)}{a(y-1) + bx}}_{(4)} \right]$$

$$- \frac{abxy}{ay + bx} \left[\underbrace{\frac{a^2 y^2 b^2 (x-1)^2}{ay + b(x-1)}}_{(3)} + \underbrace{\frac{a^3 y^3 b(x-1)}{ay + b(x-1)}}_{(2)} + \underbrace{\frac{b^3 x^3 a(y-1)}{a(y-1) + bx}}_{(1)} \right. \\ \left. + \underbrace{\frac{a^2 (y-1)^2 b^2 x^2}{a(y-1) + bx}}_{(4)} \right]$$

Combining like numbered terms gives the following expressions:

$$(1) \quad \frac{b^3 x(x-1)^2 ay}{ay + b(x-1)} - \frac{b^3 x^3 a(y-1)}{a(y-1) + bx} = ab^3 x \left[\frac{ay(y-1)(1-2x) + bx(x-1)(x-y)}{[ay + b(x-1)][a(y-1) + bx]} \right] \\ \leq O(N^2),$$

$$(2) \quad \frac{a^3 y(y-1)^2 bx}{a(y-1) + bx} - \frac{a^3 y^3 b(x-1)}{ay + b(x-1)} = a^3 by \left[\frac{bx(x-y)(1-2y) + ay(y-1)(y-x)}{[a(y-1) + bx][ay + b(x-1)]} \right] \\ \leq O(N^2).$$

Similarly,

$$(3) \quad \frac{a^2 b^2 y^2 x(x-1)}{ay + b(x-1)} - \frac{a^2 b^2 y^2 (x-1)^2}{ay + b(x-1)} = \frac{a^2 b^2 y^2 (x-1)}{ay + b(x-1)} \leq O(N^2)$$

and

$$(4) \quad \frac{a^2 b^2 x^2 y(y-1)}{a(y-1) + bx} - \frac{a^2 b^2 x^2 (y-1)^2}{a(y-1) + bx} = \frac{a^2 b^2 x^2 (y-1)}{a(y-1) + bx} \leq O(N^2).$$

Thus each of the terms is of order no more than $O(N^2)$ and so

$$|\text{Cov}(X_1^2, X_2^2)| \leq \frac{abxy}{ay + bx} [O(N^2)] \leq O(N^3).$$

We now wish to proceed inductively. Recall that we are dealing with a numbering system where X_1 represents X_i and X_2 represents X_{i+1} . The induction proof consists of assuming for some $k \geq 2$, $\text{Cov}(X_1^2, X_\ell^2) \leq O(N^3)$ for $\ell \leq k$, and then proving that $\text{Cov}(X_1^2, X_{k+1}^2) \leq O(N^3)$. The proof proceeds as follows:

Suppose once more that prior to the transition which gives increment X_1 , the force level configuration is (x, y) . Define

$S = \{\text{all possible configurations } (x', y') \text{ which can be reached from } (x, y) \text{ in } k - 1 \text{ transitions}\}.$

That is, if $(x', y') \in S$, (x', y') is a possible force level configuration immediately prior to the occurrence of X_k . Now, by assumption, $\text{Cov}(X_1^2, X_k^2) = O(N^3)$, and that covariance is given by

$$\text{Cov}(X_1^2, X_k^2) = E(X_1^2, X_k^2) - E(X_1^2)E(X_k^2).$$

But

$$\begin{aligned} E(X_1^2 X_k^2) &= E(X_1^2 X_k^2 | X_1 = -bx) P(X_1 = -bx) + E(X_1^2 X_k^2 | X_1 = ay) P(X_1 = ay) \\ &= b^2 x^2 E(X_k^2 | X_1 = -bx) P(X_1 = -bx) + a^2 y^2 E(X_k^2 | X_1 = ay) P(X_1 = ay) \\ &= \sum_{(x', y') \in S} b^2 x^2 E(X_k^2 | (x', y'), X_1 = -bx) P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\ &\quad + \sum_S a^2 y^2 E(X_k^2 | (x', y'), X_1 = ay) P[(x', y') | X_1 = ay] P(X_1 = ay), \end{aligned} \tag{A.5}$$

where $P[(x', y') | X_1] = \text{Pr}(\text{transition from } (x, y) \text{ to } (x', y') \text{ in } k-1 \text{ steps given the first step is } X_1)$. Now, $E(X_k^2 | (x', y'), X_1 = -bx)$ depends only on (x', y') as long as (x', y') is a valid point given $X_1 = -bx$, and the same holds true for $E(X_k^2 | (x', y'), X_1 = ay)$. Thus $E(X_k^2 | (x', y'), X_1 = -bx) = E(X_k^2 | (x', y'), X_1 = ay) = abx'y'$ if (x', y') can be reached after the occurrence of either value of X_1 . Therefore we may write (A.5) as

$$\begin{aligned}
E(X_1^2 X_k^2) &= \sum_S b^2 x^2 (abx' y') P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&\quad + \sum_S a^2 y^2 (abx' y') P[(x', y') | X_1 = ay] P(X_1 = ay).
\end{aligned}$$

Similarly,

$$\begin{aligned}
E(X_k^2) &= \sum_S (abx' y') P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&\quad + \sum_S (abx' y') P[(x', y') | X_1 = ay] P(X_1 = ay).
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{Cov}(X_1^2, X_k^2) &= \sum_S b^2 x^2 (abx' y') P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&\quad + \sum_S a^2 y^2 (abx' y') P[(x', y') | X_1 = ay] P(X_1 = ay) \\
&\quad - abxy \sum_S (abx' y') P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&\quad - abxy \sum_S (abx' y') P[(x', y') | X_1 = ay] P(X_1 = ay) \\
&= (b^2 x^2 - abxy) \sum_S (abx' y') P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&\quad + (a^2 y^2 - abxy) \sum_S (abx' y') P[(x', y') | X_1 = ay] P(X_1 = ay).
\end{aligned}$$

A similar approach may be used to calculate $\text{Cov}(X_1^2, X_{k+1}^2)$, conditioning once again on the points (x', y') which are possible force level configurations prior to the occurrence of X_k . This procedure gives

$$\begin{aligned}
\text{Cov}(X_1^2, X_{k+1}^2) &= (b^2 x^2 - abxy) \sum_S \left[abx'y' - \frac{ab^2 x'^2 + a^2 by'^2}{ay' + bx'} \right] \cdot \\
&\quad P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&+ (a^2 y^2 - abxy) \sum_S \left[abx'y' - \frac{ab^2 x'^2 + a^2 by'^2}{ay' + bx'} \right] P[(x', y') | X_1 = ay] \cdot \\
&\quad P(X_1 = ay).
\end{aligned}$$

It follows that

$$\begin{aligned}
\text{Cov}(X_1^2, X_{k+1}^2) &= \text{Cov}(X_1^2, X_k^2) - \sum_S (bx^2 - abxy) \left[\frac{ab^2 x'^2 + a^2 by'^2}{ay' + bx'} \right] \cdot \\
&\quad P[(x', y') | X_1 = -bx] P(X_1 = -bx) \\
&- \sum_S (a^2 y^2 - abxy) \left[\frac{ab^2 x'^2 + a^2 by'^2}{ay' + bx'} \right] P[(x', y') | X_1 = ay] P(X_1 = ay).
\end{aligned}$$

Both of the sums in the above expression are of the form of expected values of functions which are at most $O(N^3)$ and, since $\text{Cov}(X_1^2, X_k^2) \leq O(N^3)$, it follows that

$$|\text{Cov}(X_1^2, X_{k+1}^2)| \leq O(N^3).$$

Thus, by induction, $\text{Cov}(X_1^2, X_l^2) \leq O(N^3)$ for all $l > 1$. Now recall that X_1 represented X_i for any i , and that the force level configuration (x, y) was also arbitrary. Thus the proof holds in general and $\text{Cov}(X_i^2, X_j^2) \leq O(N^3)$ for all i and j .

In this case, $\sum_i \sum_j \text{Cov}(X_i^2, X_j^2)$ must be of order no greater than $O(N^5)$ and Lemma 3 is established.

It follows from Lemma 3 that

$$\text{Var} \left(\sum_{k=1}^{N_n} x_k^2(n) \right) \leq O(N^5)$$

and so

$$\frac{\text{Var} \left(\sum_{k=1}^{N_n} x_k^2(n) \right)}{s_{N_n}^4(n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and the proof of Lemma 2 is complete.

We have, by Chebyshev's inequality, and Lemma 2,

$$P(|s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n) - 1| > \epsilon) \leq \text{Var}[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n)] / \epsilon^2 \rightarrow 0$$

and so $s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n) \xrightarrow{P} 1$. This completes the proof of Lemma 1.

With Lemma 1 verified, we now wish to demonstrate:

Lemma 4.

$$\lim_{n \rightarrow \infty} s_{N_n}^{-2}(n) \sup_{k \leq N_n} E(x_k^2(n)) \leq \lim_{n \rightarrow \infty} E[s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n)] = 0. \quad (\text{A.6})$$

Proof: The proof employs expression (A.2) and a lemma due to Pratt (1960).

Lemma - Pratt. If f_n, f, g_n, g, G_n, G are measurable functions on a measure space $(\Omega, \mathcal{B}, \mu)$ and if

- (i) $f_n \rightarrow f, g_n \rightarrow g, G_n \rightarrow G$ in measure,
- (ii) $g_n \leq f_n \leq G_n \quad \forall n$

(iii) $\int g_n dm \rightarrow \int g dm$, $\int G_n dm \rightarrow \int G dm$ as $n \rightarrow \infty$ with
 $\int g dm$ and $\int G dm < \infty$, then $\int f_n dm \rightarrow \int f dm$ if
 $\int f dm$ is finite.

Now,

$$0 \leq s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n) \leq s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n).$$

Relating this expression to Pratt's Lemma, let $g_n \equiv 0 \quad \forall n$,

$f_n = s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n)$ and $G_n = s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n)$. Clearly

$g_n \rightarrow g \equiv 0$, while from (A.3) $G_n \rightarrow G = 1$, and from (A.2)

$f_n \rightarrow f = 0$. Also $\int g_n dm \equiv 0 \equiv \int g dm \quad \forall n$ and

$$\int G_n dm = E[s_{N_n}^{-2}(n) \sum_{k=1}^{N_n} x_k^2(n)] = 1 \quad \forall n.$$

(Note that the integration is with respect to dP , the appropriate probability measure over the space.)

Thus, by Pratt's Lemma

$$E[s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n)] = \int f_n dm \rightarrow \int f dm = 0.$$

But $s_{N_n}^{-2}(n) \sup_{k \leq N_n} E(x_k^2(n)) \leq E[s_{N_n}^{-2}(n) \sup_{k \leq N_n} x_k^2(n)] \rightarrow 0$ and so Lemma

4 is verified.

The next result needed to complete the demonstration of (A.1) is:

Lemma 5.

$$s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) \xrightarrow{P} t.$$

Proof: To prove this lemma we show

$$|s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) - t| \leq s_{N_n}^{-2}(n) |E(X_{m_n(t)+1}^2(n))| \rightarrow 0. \quad (A.7)$$

First, from the definition of $m_n(t)$ given above

$$m_n(t) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

Now, $|s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) - t| = s_{N_n}^{-2}(n) |s_{m_n(t)}^2(n) - t s_{N_n}^2(n)|$. But,

by the definition of $m_n(t)$, $s_{m_n(t)}^2(n) \leq t s_{N_n}^2(n)$ while

$s_{m_n(t)+1}^2(n) > t s_{N_n}^2(n)$. Thus

$$\begin{aligned} |s_{m_n(t)}^2(n) - t s_{N_n}^2(n)| &\leq |s_{m_n(t)}^2(n) - s_{m_n(t)+1}^2(n)| \\ &= |E(X_{m_n(t)+1}^2(n))|. \end{aligned}$$

Therefore,

$$\begin{aligned} |s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) - t| &\leq s_{N_n}^{-2}(n) E(X_{m_n(t)+1}^2(n)) \\ &\leq s_{N_n}^{-2}(1) \sup_{k \leq N_n} E(X_k^2(n)) \xrightarrow{P} 0, \end{aligned}$$

and so $s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) \xrightarrow{P} t$.

With Lemma 5 proved, the demonstration of (A.1) is now completed by showing

$$s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{P} t.$$

Consider, therefore,

$$\begin{aligned} & |s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) - s_{N_n}^{-2}(n) s_{m_n(t)}^2(n)| \\ &= s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) |s_{m_n(t)}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) - 1|. \end{aligned} \quad (A.8)$$

It follows that

$$s_{m_n(t)}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{P} 1 \text{ while } s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) \xrightarrow{P} t$$

from Lemma 5, since $m_n(t) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, the right-hand side of (A.8) converges in probability to zero and so the left-hand side must also converge to zero in probability. That is,

$$s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) - s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) \xrightarrow{P} 0.$$

Since $s_{N_n}^{-2}(n) s_{m_n(t)}^2(n) \xrightarrow{P} t$, it follows that

$$s_{N_n}^{-2}(n) \sum_{k=1}^{m_n(t)} x_k^2(n) \xrightarrow{P} t,$$

and condition (A.1) is verified. Thus Watson's Square Law martingale satisfies Scott's central limit theorem.

Note that after the proof of expression (A.3), the actual form of the martingale never again plays a direct role in the demonstration. In fact, the remainder of the above proof is merely a generalization of Scott's results to triangular arrays with N_n elements in the n th row. The fact that the martingale proposed in Watson (1976) for use with a Lanchester Square Law combat model satisfies expressions (A.2) and (A.3) insures that a triangular array formed from this martingale in the manner described will satisfy the martingale central limit theorem of Scott (1972).

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